Estimation of Response Surface Gradients in Multiobjective Water Resources Planning

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A probabilistic Markov model of reservoir state transitions is developed which, in combination with a simulation model, offers a computational tool for estimating response surface gradients of water resources planning problems. Use of the gradient estimation capability provides a substantial improvement in efficiency of the search for improved water resources configurations. The estimation capability arises by recognizing and using the reservoir state transition changes that would result from a modification of decision variables. The procedure is highly flexible in that it functions under such complicated situations as multiple-objective environments, nonlinear benefit loss functions, and complex connectivity of river basin components.

The increasing demands being placed upon water resources developments are requiring planners to identify allocations of water between many, often conflicting, purposes. Plan selection must include the determination of the set of sizes and operating policies for all reservoirs, irrigation areas, diversions, hydroelectric power installations, etc., which is technically and politically feasible and which by some measure is better than all alternative solutions. For a complex river basin, with many interest groups and many potential development alternatives, there may be a large number of feasible solutions, and consequently the determination of the 'best' solutions or configurations is a very difficult problem.

To create an environment for analyzing the feasible alternatives, recourse has been made to systems analysis through the use of mathematical models. However, successful application of the available models has been seriously hampered by the absence of an efficient means for estimating the gradient vector of the response surface. This paper describes a model that fills this need. The response surface gradient information provided by the model will be seen to facilitate a much more rational selection of solutions with improved responses and, in turn, to provide a substantial improvement in efficiency of the search for improved water resources configurations.

A MULTIPLICITY OF MODELS

Within the general basin-planning problem for surface water developments, experience has demonstrated that no single mathematical model is capable of simultaneously capturing a complete description of the whole problem while maintaining a closed-form analytic formulation. As a result, the usual procedure has been to create several mathematical models, each constructed to examine selected elements of the full problem with the net accumulated information eventually identifying the best plan of development. The sophistication of the mathematical models that have been employed in this task varies greatly, the complexity being a function of the problem addressed. Nevertheless, a generalization of the mathematical models developed in the water resources field to date might segment them into (1) screening models, (2) simulation models, and (3) other models such as sequencing models, dynamic policy models, econometric and accounting models. This paper is concerned solely with screening and simulation models.

The worth of screening models arises since they both incorporate quantitative relationships between system variables and generate optimal solutions as measured by an explicit objective ranking of alternatives. Because of their analytic, closed-form characteristics, screening models frequently possess the capability for generating many solutions for a small price. However, an important shortcoming of screening models is the necessary set of assumptions and approximations that is required to obtain an analytic form. Conversely, the second group of mathematical models, simulation models, are more flexible than screening models since they consist of a sequence of mathematical and logical statements describing the size and operation of the components of the system being modeled.

Simulation models have not been an effective means for choosing or defining the optimum solutions because they do not decide what should be done to achieve a given objective or objectives. They allow only the evaluation of the economic or physical response of the system to a particular set of decision variables. Thus each run of the simulation model maps only one set of decision variables into an economic response. In principle, it should be possible to search the response surface for the optimal configuration of decision variables. In practice, this is almost never done because current procedures for indicating how the search process should occur are inefficient.

This brief discussion has demonstrated that neither screening models nor simulation models offer, separately, a viable capability to locate the optimal configurations. This suggests that the role of systems analysis models in the planning process should begin with the use of a screening model to locate good feasible solutions and should be followed by an examination of these and similar solutions by the simulation model. This approach has recently begun to receive widespread acceptance [e.g., *Grayman et al.*, 1971; *McBean and Schaake*, 1973; *Jacoby and Loucks*, 1972].

The determination of similar solutions for analysis by the simulation model is an important element in the search for the optimal configurations. The intent in examining successive solutions is to locate solutions with improved responses. A successive trial-and-error search for solutions with improved responses would be highly inefficient for a complex, multidimensional system; there is a need for an analytic framework to estimate the gradient of the response surface in the region of the current set of decision variables. Response surface gradient information would facilitate a much more rational selection of solutions with improved responses.

To derive the theoretical basis upon which a gradient estimator will be structured, it is useful to reflect upon certain probabilistic models of reservoir operation that have been proposed in the literature. Notable early contributions of probabilistic models were made by *Moran* [1955, 1959], *Prabhu* [1958], *Gani* [1957], *Gani and Moran* [1955], and *Langbein* [1958]. These early analyses of the probabilistic behavior of reservoirs attempted to work with the probability density function for storage levels, reservoir inflows, and reservoir outflows; but the analytical complexities of the problem prevented the development of these techniques beyond the most elementary stages. Later investigators recognized that the Monte Carlo techniques offered a simpler computational tool for these complex dam problems, which consequently slowed the pursuit of the probabilistic formulations. The result is that the practical utility of the probabilistic models has never really been established.

Nevertheless, in a different application to the same problem (i.e., in the search to determine the gradient of the response surface) these early theoretical contributions offer a useful framework. In other words, the prospect will be developed of returning to the analytical statistical basis for a simulation model to find a way to use the simulation results. It will then be shown how a technique, called the 'state-augmented Markov analog,' is particularly well adapted to estimate the gradient of the response surface.

The Gradient Estimator

To establish the basis for the gradient estimator, it is first necessary to briefly review specific elements of the probabilistic models. Assume that the reservoir storage volume at a reservoir is discretized into N segments ΔS_j, where $j = 1, \cdots, N$ and the jth storage segment is defined as $\Delta S_j = S_j - S_{j-1}$. Given the marginal probability distribution of flows into the reservoir during season i, $f(X_i)$, and the operating policy for the reservoir during season i, there is some conditional probability P_{idh} that the reservoir, if it currently occupies state d, will occupy state h after the transition of the ith season. The set of conditional probabilities P_{idh} over all d and h form the storage transition matrix P_i.

Multiplication of the transition matrix $\mathsf{P}_1\mathsf{P}_2 \cdots \mathsf{P}_{N_s}\mathsf{P}_1 \cdots$ demonstrates a continuous progression toward a matrix with identical rows [*Hillier and Lieberman*, 1967]. The net outcome, if it is assumed that the system is stationary (i.e., the demands are not growing in time and the streamflows are not more than a function of season), is the development of a terminal steady state storage density function giving the vector of probabilities $\mathsf{P}(1 \times N)$ of the reservoir being in any state $j, j = 1, \cdots, N$, at the end of a specified time period.

Since the model implies a dependence strictly on only the previous storage level (i.e., the probability P_{idh} of being in a state h is a function of d and not of previous storage levels), the assumption is implicit that successive storage levels behave as a single-lag Markov chain and that successive hydrologic inflows are independent. This assumption of independence of successive inflows represents a principal reason for the lack of widespread acceptance of the probabilistic models.

More discussion of the effects of the independence assumption is delayed to a later section; a characteristic of more immediate concern is that the computed results from the model specify Dirac delta functions at the zero and full states of the reservoir to (1) deliver the release demand and (2) store excess water in the reservoir for later use. Note that no information is available from the model to indicate the degree of failure; e.g., the severity of the shortfall of the release is unknown.

To demonstrate how the model may be extended to provide information regarding the severity of a failure, it is useful to assume for discussion purposes that the reservoir is to be operated in accord with the 'normal operating rule,' the essential features of which are documented in Figure 1. Consider now a new storage variable S' that may take on values less than zero or greater than the reservoir capacity S_m such that the reservoir release is always equal to the release target Q_i, where i denotes the season. S' is consequently defined over the range $-\infty < S' < \infty$ and represents an augmented form of S.

Associated with transitions of S' from season to season is a P_{idh}' which identifies the probability of storage state transitions from d to h, where d and h are now defined over N', where $N' > N$.

A point that is worthy of clarification is the situation where h, for example, corresponds to an S' that is $>S_m$ or <0. It is mandatory that a transition into and out of each state of reservoir storage S' be possible, and yet 'negative storage' is a physical impossibility and only a mathematical rubric for identifying the system shortfall. Therefore if a transition d to h' occurs in season i (where h' corresponds to $S' < 0$), then the transition from h' to some subsequent level h'' in season $i + 1$ assumes an initial storage level of zero. By such an action, physical continuity is not violated in the reservoir transitions, and yet information is retained that indicates the degree of failure.

State transitions in the augmented transition matrix P_i', where $\mathsf{P}_i' = N' \times N'$, are basically the same as those of P_i, only now instead of a transition to the zero storage state when the demands Q_i exceed available water, the transition is made to a negative storage. The terminal state h assumes a level indicating the magnitude by which the actual release Y failed to meet the target Q_i (measured in storage units). Association with a short-term loss function [e.g., *Loucks*, 1969] provides the information necessary to attach a short-term benefit level to any state in the augmented terminal steady state storage density function $\mathbf{P}'$ $(1 \times N')$.

The state-augmented form of the transition matrix is illustrated in Figure 2, where, for example, a transition from j'' to d' implies a failure to provide the target release. The state-augmented form of the transition matrix provides estimates of the augmented, terminal steady state storage density function $f(S_i')$ and, in turn, from continuity, estimates of the probabil-

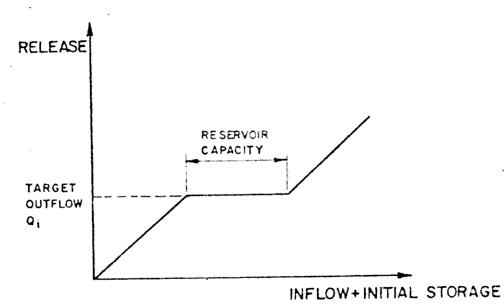

Fig. 1. Normal operating rule.

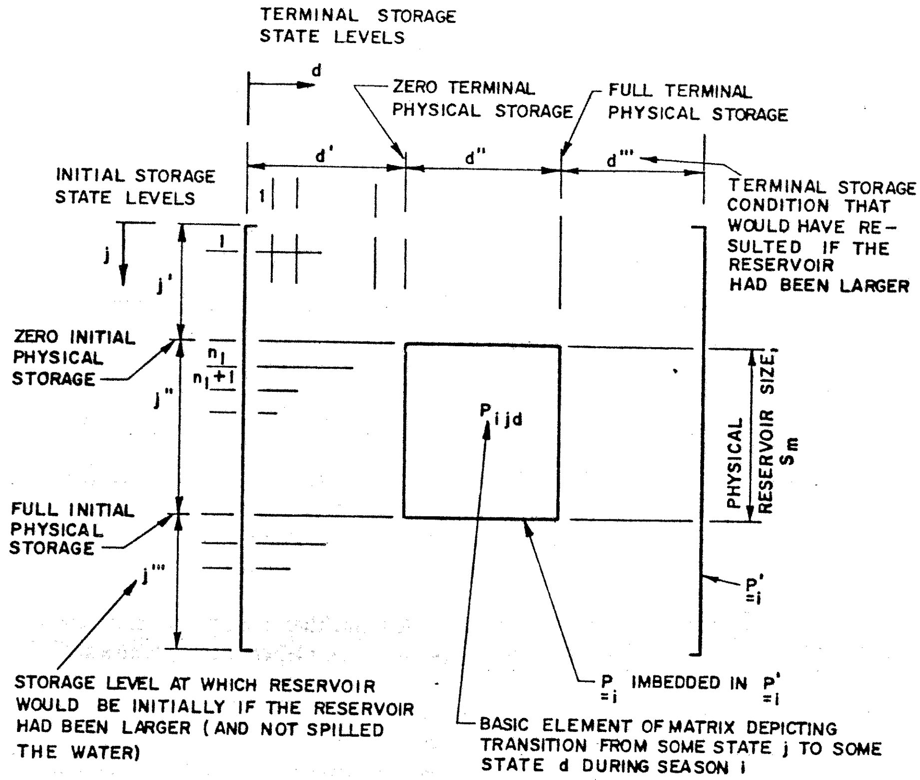

Fig. 2. State-augmented form of the transition matrix.

ity function of release $f(Y_i)$. The $f(Y_i)$ function can be related to the benefits obtained at downstream use points.

It is important to note that the transition matrices P_i' are representative of the reservoir transition behavior only for a particular set of decision variables (i.e., for a specified reservoir capacity S_m and release policy Q_i). There are, however, two interesting features of the transition matrices P_i': (1) they retain a significant amount of the information generated during the simulation run (accomplished by using the simulation model during execution to identify the state transitions that are entries in the matrices, the assumption being that the state transitions identified by the simulation model are based on the augmented storage vector described above); and (2) they are shown to be amenable to modification to reflect the change in system response associated with a marginally adjusted set of decision variables. As a result, P_i' can be made to provide the information necessary to examine the gradient of the response surface.

To examine the mechanism for the estimation of the gradient response, assume that the objective function is the maximization of the weighted sum of expected net benefits, so that, consistent with the normal operating rule, the weighted sum of expected benefits during season i can be written as a function of the available water, i.e.,

$$E[B_i] = \sum_{m=1}^{M} W_m \int_S \int_X f_i(S, X) \cdot B_{im}(S, X)\, dS\, dX \qquad (1)$$

where M is the number of objectives under consideration, W_m is the weight associated with the mth objective (for a discussion of 'weight' selection, see *Cohon and Marks* [1973]), $f_i(S, X)$ is the joint density function of S and X for season i, $B_{im}(S, X)$ is the conditional benefit response to the mth objective given the levels of X and S, and X is the hydrologic inflow to the reservoir. Evaluation of the integration in (1) is the natural function of the simulation model during its execution. The derivation of $f_i(S, X)$ from the density functions for the streamflow variables involves a large number of integrations which are implicit in the simulation model.

As a first step to develop the response surface gradient, the effects of X may be integrated out of (1) by use of the simulation model to leave

$$E[\langle B_i\rangle] = \sum_{m=1}^{M} W_m \int_S \langle B_{im}\rangle(S) f_i(S)\, dS \qquad (2)$$

where the angle brackets around B_i indicate that it is an average created by integrating over all X_{it}, where t runs from 1 to T, T being the length of the simulation. In a finite state equivalent, (2) can be written as

$$E[\langle B_i\rangle] = \sum_{m=1}^{M} W_m \sum_{j=1}^{N'} P_{ij}'\langle B_{im}\rangle(B|j) \qquad (3)$$

where P_{ij}' is the terminal, or end-of-time period, steady state storage distribution for season i and $\langle B_{im}\rangle(B|j)$ is the expected benefit B during season i, conditional on the state-augmented storage being in state j.

Rewriting (3) in vector form gives

$$E[\langle B_i\rangle] = \mathbf{W}\langle \mathbf{B}_i\rangle^T \mathbf{P}_i' \qquad (4)$$

where $\langle \mathbf{B}_i\rangle^T$ denotes the transpose of $\langle \mathbf{B}_i\rangle$.

Assume that the set of decision variables subject to potential marginal adjustment is summarized in some vector $\mathbf{Z}$ as per

$$E[\langle B_i\rangle(\mathbf{Z})] = \mathbf{W}\langle \mathbf{B}_i\rangle^T(Z)\mathbf{P}_i'(\mathbf{Z}) \qquad (5)$$

The expected benefits computed from (5) will provide a value identical to the value that is computed by the simulation model (if the simulation results were discretized in a similar fashion). To evaluate the gradient of the response surface in the region of Z, the problem is to derive the change in weighted sum of expected net benefits for some $\mathbf{Z} + \Delta\mathbf{Z}$. The marginal change in the expected benefits may be written as

$$E[\langle \hat{B}_i \rangle(\mathbf{Z} + \Delta\mathbf{Z})] = \mathbf{W}\langle \hat{\mathbf{B}}_i \rangle^T(\mathbf{Z} + \Delta\mathbf{Z}) \cdot \hat{\mathbf{P}}_i'(\mathbf{Z} + \Delta\mathbf{Z}) \quad (6)$$

where a circumflex is employed since the value is only an estimate. Two of the right-hand side terms can be written as

$$\langle \hat{\mathbf{B}}_i \rangle^T(\mathbf{Z} + \Delta\mathbf{Z}) = \langle \mathbf{B}_i \rangle^T(\mathbf{Z}) + \Delta\langle \hat{\mathbf{B}}_i \rangle^T(\mathbf{Z} + \Delta\mathbf{Z}) \quad (7)$$

and

$$\hat{\mathbf{P}}_i'(\mathbf{Z} + \Delta\mathbf{Z}) = \mathbf{P}_i'(\mathbf{Z}) + \Delta\hat{\mathbf{P}}_i'(\mathbf{Z} + \Delta\mathbf{Z}) \quad (8)$$

By substituting (7) and (8) into (6) and neglecting the higher-order product of the two difference terms (which argues for keeping ΔZ small), the change in weighted sum of expected benefits can be written as

$$E[\Delta\langle \hat{B}_i \rangle] = \mathbf{W} \cdot [\Delta\langle \hat{\mathbf{B}}_i \rangle^T(\mathbf{Z} + \Delta\mathbf{Z}) \cdot \mathbf{P}_i'(\mathbf{Z}) + \langle \mathbf{B}_i \rangle^T(\mathbf{Z}) \cdot \Delta\hat{\mathbf{P}}_i'(\mathbf{Z} + \Delta\mathbf{Z})] \quad (9)$$

Similarly, the marginal change in costs (e.g., due to adjustment in reservoir capacity) can be computed as the difference in the equivalent annual cost level Δc as

$$\Delta c = \mathbf{W} \cdot [c(\mathbf{Z} + \Delta\mathbf{Z}) - c(\mathbf{Z})] \quad (10)$$

and so the change in weighted sum of expected net benefits can be written as

$$E[\Delta N\langle \hat{B}_i \rangle] = \sum_{i=1}^{N_t} E[\Delta\langle \hat{B}_i \rangle] - \Delta c \quad (11)$$

One alternative for estimating $E[\Delta N\langle \hat{B}_i \rangle]$ would be to employ a univariate search à la *Beard* [1967], where the procedure would involve use of the simulation model for Z and then for $\mathbf{Z} + \Delta\mathbf{Z}$. This procedure would provide the necessary information but only at the price of substantial computational expenditure. An alternative, however, is to utilize the simulation model for Z and, during execution, to construct the transition matrices $\mathbf{P}_i'$. The discussion below will identify how $\mathbf{P}_i'$ may then be adjusted to reflect the marginally adjusted set of decision variables $\mathbf{Z} + \Delta\mathbf{Z}$ and to allow computation of the modified weighted sum of the change in expected net benefits of (11).

Implementation of these ideas can be traced as follows. The simulation model is utilized in a normal simulation context for Z but with the additional characteristic that it simultaneously integrates over X_{it} for all i, t to provide $\mathbf{P}_i'$ for all i for Z. For evaluation of $\mathbf{Z} + \Delta\mathbf{Z}$ it is possible to modify $\mathbf{P}_i'$ for all i matrices to develop estimates of $f_i(S')$ corresponding to the modified set of decision variables. A series of simple algorithms can be programed into the computer to appropriately modify $\mathbf{P}_i'$ for, for example, (1) an incremental adjustment of release level Q_i to $Q_i - \delta Q_i$ (e.g., to reflect a decreased size of a downstream demand point) and (2) an incremental adjustment of reservoir capacity S_m to $S_m + \delta S_m$, which can be handled by reflecting the effect of a modified size of $\mathbf{P}_i$ within $\mathbf{P}_i'$. The release adjustment would change the probability of a transition from a state j to a state d to a different transition, say, j to d', where $d' > d$ (i.e., the reservoir would contain a larger amount of water in storage at the end of the time period due to the decreased demand level). (A range of policies more complex than the normal operating rule would experience a more involved adjustment to the transition matrix but may be handled. An expanded discussion is given by *McBean and Schaake* [1973].)

Given the adjusted form of the transition matrix, the storage density function for the revised set of decision variables is generated, which is an important element in determination of the change in the weighted sum of expected net benefits as computed by $E[\Delta N\langle \hat{B}_i \rangle]$ in (11). The evaluations of the other terms in (11) (i.e., the conditional benefit function and the change in conditional benefit function) are necessarily problem dependent but are available from the simulation model.

The determination of $E[\Delta N\langle B_i \rangle]$ provides an estimate of the magnitude of the gradient of the response surface in one vector direction in the region of the current set of decision variables. The gradient information over n directions, where n denotes the total number of variables susceptible to potential adjustment, can be accomplished by similar adjustments and application of the analog. Note that with use of the analog the simulation model need not be rerun to determine the best direction of movement; instead only interesting subanalyses employing the analog need be done.

The use of the analog in conjunction with a simulation model is summarized in Figure 3.

Introduction to an Example Problem

To test the abilities of the analog in an application, a simple two-season basin development problem of a reservoir and downstream irrigation area was created to isolate many of the interesting issues. The hydrologic inflow to the system, X, was purposely chosen to relate to real world experience. A historical time series of 42 years of hydrologic flows was selected from the Pichi Mahuida gaging station on the Rio Colorado, Argentina. Synthetically generated flows were used in the subsequent analyses.

The reservoir will be assumed to follow the normal operating rule, where it may, if required, be constructed to store water in excess of current irrigation requirements for use later when the supply of water in the stream is less than the irrigation requirements. The irrigation requirements are highly seasonal, the first season corresponding to winter (and con-

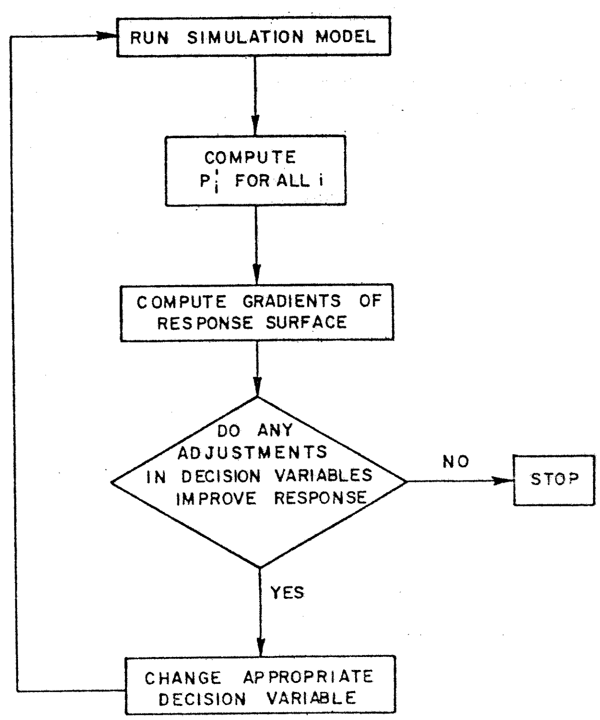

Fig. 3. Summary of analog use in conjunction with the simulation model.

sequently irrigation flow requirements being small and taken as zero) and the second season being the growing season.

Assume that the objectives of interest are limited to national income and regional income. National income benefits and costs are to be determined in accord with the usual procedures for the national income account [*Major,* 1969]. The regional costs and benefits depend upon the cost-sharing arrangements; assume that the region will receive 75% of the irrigation benefits and will be required to pay only 25% of the dam costs. Accordingly, national income net benefits and regional income net benefits may be computed for each alternative.

A simple deterministic continuous screening model was used to provide initial estimates of the transformation surface, as is illustrated in Figure 4. Upon experimentation with the simulation model the net benefits for these alternatives were found to be much less than predicted by the screening model. This suggests that other alternatives exist which may give more reliable system operation and, possibly, greater net benefits than the alternatives proposed by the screening model.

Applying the simulation model to the design alternatives suggested by the screening model indicated the results depicted in Figure 5. The divergence between net benefits as computed by the simulation and screening models is very sizable indeed.

The difficulty is now one of determining the appropriate adjustments in decision variables to improve the system performance. To create a forum for the successive marginal analysis procedure, the response surfaces for a series of different weighting functions were analyzed on a uniform grid basis over the range of interesting sets of alternatives by the simulation model. The analog and search procedures were then applied to a set of weighted objective functions. Two examples of this successive search procedure are indicated in Figures 6 and 7.

The variable levels corresponding to the optimal solution for each set of weights are seen to be considerably different from those suggested by the screening model. For example, in weight set b (Figure 6) the maximum net benefit ϕ' occurs for an alternative [A'] in which no dam is built, but the size of the irrigation area is smaller than in alternative [A]. This shows that greater net benefits can be obtained from more reliable operation of the irrigated area. Further, for weight set c (Figure 7) the variable set moved the complete spectrum from a very large reservoir to no reservoir construction at all.

The optimal solutions from the simulation model for the various weights are points on the net benefit transformation surface. If the range of weights are assumed to be sufficiently varied, the points on the surface are sufficient markers to allow curves to be traced through to describe the complete net benefit transformation surfaces as seen by the screening model and the simulation model, the latter being a good estimate of the true surface. The simulation model net benefit transformation surface is presented in Figure 8.

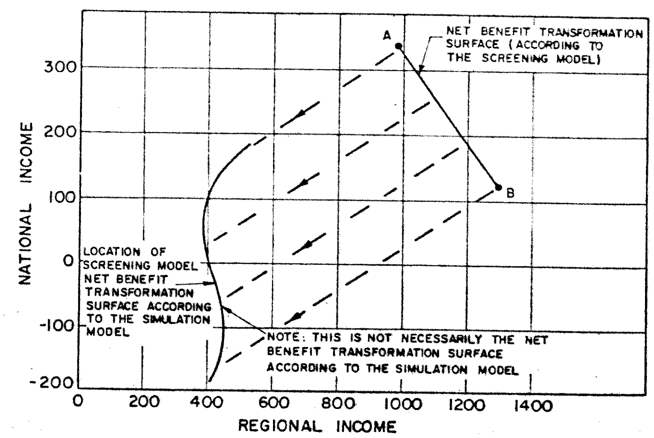

Fig. 5. Simulation model evaluation of the alternatives on the screening model net benefit transformation surface.

MARGINAL ANALYSIS OF A MULTIRESERVOIR SYSTEM

The analytic capabilities discussed previously are useful in analyzing the marginal analysis questions at only a single reservoir and the decision variables associated with that reservoir (e.g., the release level Q_i which reflects the needs of an irrigation area downstream of the reservoir). The practical usefulness of the analog depends, however, on whether the gradient estimation procedure is applicable to complex basin problems in which there are a series of reservoirs and use points. The principal complicating aspect of complex basins enters because the inflow pattern to a reservoir is a consequence of any upstream flow regulation. This regulation depends on the operation and capacity of further upstream

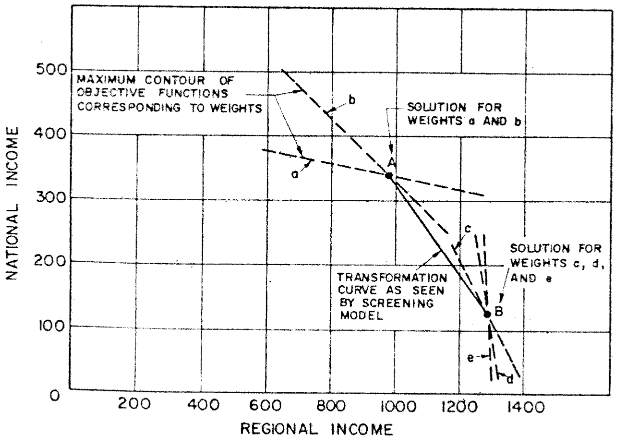

Fig. 4. Net benefit transformation curve, as seen by the screening model.

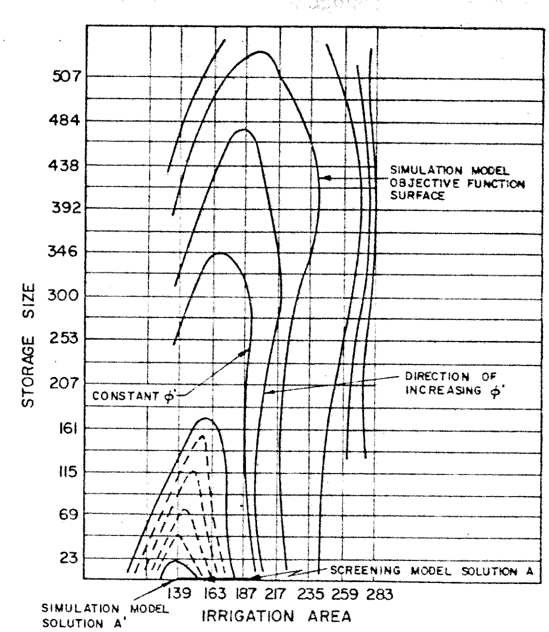

Fig. 6. Sequential search mechanism over the response surface (weight set b).

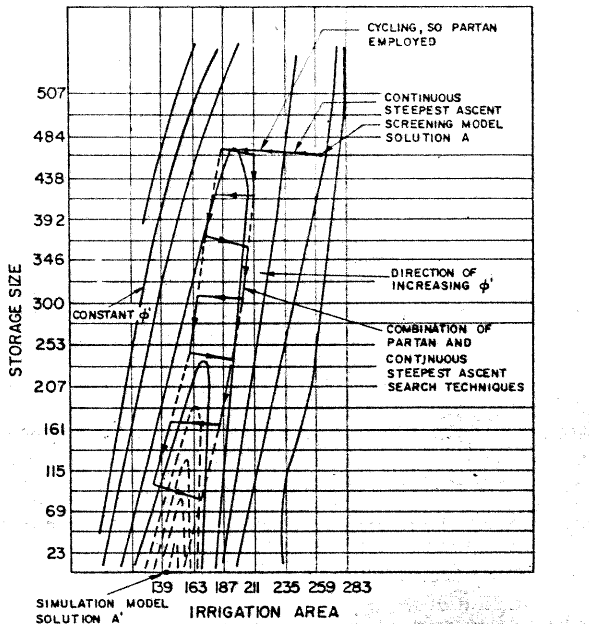

Fig. 7. Sequential search mechanism over the response surface (weight set c).

reservoirs, the withdrawal demands for intermediary points of use (e.g., irrigation areas), and the allocation policy for water distribution.

The necessary extension to apply the analog to the complex basin problem is the recognition of the connectivity effects that adjustments in decision variables at a reservoir have in terms of modifying the inflow distribution to downstream reservoirs. The discussion below will indicate how the analog can reflect many of the important elements in this interconnectivity without either an excessive computational burden or excessive storage requirements when it is applied to the complex basin problem. The exponential increases in computation and storage experienced with screening models [*Loucks*, 1969] are avoided in the analog because the complex system can be broken into discrete segments.

To establish a notational convention, assume that a complex basin consists of K reservoirs connected in series, where the reservoirs are numbered $k = 1, 2, \cdots, K$ by starting at the bottom of the system. Associated with each reservoir k there are assumed to be a number of use points UP_k (where $0 \leq UP_k$) that are downstream from k but upstream from $k - 1$. The kth reservoir and its UP_k are classified henceforth as the kth segment. The extension of the conditional benefit response computation from the single downstream use point of the previous section to include the effects of all the use points downstream from k but upstream from $k - 1$ is straightforward once an allocation policy is established.

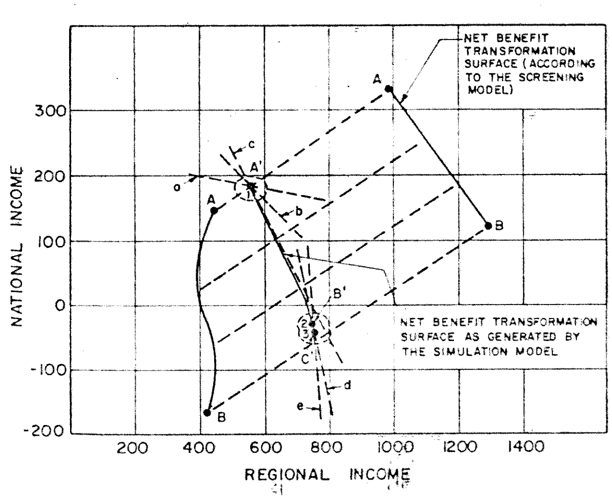

Fig. 8. Simulation model net benefit transformation surface.

The need for an allocation policy arises when the sum of target demands exceeds the supply and a policy for distributing the available water between the demand points is required. There are many possible forms of the allocation policy (e.g., it can equalize the deficit or percentage of deficits to all points, allocate the water to locations with the highest marginal benefits, or allocate the water to uses on the basis of a system of priorities or water rights seniorities). The allocation policy must reflect the demands downstream of $k - 1$, since a benefit is associated with use of the release from k that becomes inflow to $k - 1$. As this is, in essence, a transfer of water from segment k to segment $k - 1$, the expected benefit associated with the downstream use of the inflow to $k - 1$ will be considered as a transfer payment and denoted by T_k.

A generalization of the weighted sum of expected benefit computation for the kth segment becomes, in vector notation,

$$E[B_{ki}] = \mathbf{W}\langle\mathbf{B}_{ki}\rangle^T \cdot \mathbf{P}_{ki} + \mathbf{W} \cdot \mathbf{T}_{ki} \tag{12}$$

where $\mathbf{T}_{ki}$ in its present form is the expected benefits of downstream use of the water released from k during season i and the other terms are as used above. To demonstrate the evaluation of (12), consider the following example discussion of an adjustment in decision variable δQ_{ki} at reservoir k. The marginal change in benefits at the use points UP_k can be evaluated as discussed earlier, given the allocation policy. The marginal change in transfer benefit $\Delta\mathbf{T}_{ki}$ can be evaluated by recognizing the altered inflow pattern to $k - 1$ and therefore the adjustment in available water for meeting targets downstream of $k - 1$. If an incremental change in release policy in season i, δQ_{ki}, simply created an adjusted inflow to $k - 1$ of some quantity $\delta X_{k-1,i}$, evaluation of $\Delta\mathbf{T}_{ki}$ would be relatively simple, since it would equal the marginal value of an additional increment of water. This approach to the problem would be incorrect because hydrologic variability creates shortages and surpluses in the delivered flow, the implication being that δQ_{ki} is not always released. However, the appropriate effects can be captured within the analog for a change δQ_{ki}, since the modifications in terminal storage density function indicate the modified pattern of actual releases due to the change in release policy. Thus when $\mathbf{P}_{ki}'$ is modified to reflect δQ_{ki}, the modified release pattern is available. This information, in combination with the allocation policy, allows computation of the adjusted inflow pattern to $k - 1$.

To translate the adjustment effects to $\Delta\mathbf{T}_{ki}$, consider the response at $k - 1$. In continuous form, (1) related the expected benefits at $k - 1$ as

$$E[B_{k-1}] = \int_X \int_S B_{k-1}(S, X) f(S_{k-1}, X_{k-1})\, dS\, dX \tag{13}$$

(The subscript $k - 1$ has now been retained to associate the benefits with the $k - 1$ segment.) Instead of being used to integrate out the effects of X, the simulation model could be utilized to integrate out the effects of S to leave

$$E[\langle\langle B_{k-1}\rangle\rangle] = \int_X \langle\langle B_{k-1}\rangle\rangle(X) f(X_{k-1})\, dX \tag{14}$$

where $\langle\langle B_{k-1}\rangle\rangle(X)$ now denotes the conditional annual benefit given the level X_{k-1} and the double angle brackets around $\langle\langle B_{k-1}\rangle\rangle$ arises because the averaging process has been invoked for all S.

In discrete form, (14) becomes

$$E[\langle\langle B_{k-1}\rangle\rangle] = \mathbf{W}\langle\langle \mathbf{B}_{k-1}\rangle\rangle^T \mathbf{P}_{k-1}' \quad (15)$$

where $\mathbf{P}_{k-1}'$ represents the discrete form of the marginal density of inflows to reservoir $k - 1$ and $\langle\langle \mathbf{B}_{k-1}\rangle\rangle$ represents the discrete form of the benefit that is conditional on the level of inflow. The marginal net benefit change for the ith season, the higher-order product of the two difference terms being neglected, becomes

$$E[\Delta\langle\langle B_{k-1,i}\rangle\rangle] = \mathbf{W}\cdot\Delta\langle\langle \hat{\mathbf{B}}_{k-1,i}\rangle\rangle^T \mathbf{P}_{k-1,i}' + \mathbf{W}\cdot\langle\langle \mathbf{B}_{k-1,i}\rangle\rangle^T\cdot\Delta\hat{\mathbf{P}}_{k-1,i}' \quad (16)$$

In a manner similar to that used for the modification of $\mathbf{P}_{k-1,i}'$ to reflect adjustments in the release policy and the storage size, it is equally possible to modify $\mathbf{P}_{k-1,i}$ to reflect an adjusted inflow pattern. If independence between any $S_{k-1,t}$ and $X_{k-1,t+1}$ is assumed (as is implicit in the analog), an adjustment $\Delta P_{k-1,i,j,d}$ to $P_{k-1,i,j,d}'$ is identical to the appropriate modification $\Delta P_{k-1,i,j+1,d+1}$ to $P_{k-1,i,j+1,d+1}'$ (assuming that the sizes $\Delta S_{k-1,ij}'$ and $\Delta S_{k-1,ij+1}'$ are identical). (The assumption of independence, i.e., $f(S, X) = f(X)f(S)$, is not essential if the nature of the correlation can be translated into the effect of modifying X_{k-1} at various levels of S_{k-1}'. This assumption, however, represents an algorithm that can be easily incorporated into an internal set of computer program commands.) The above result follows directly from the continuity equation.

The modified transition matrix creates the corresponding altered terminal storage condition at $k - 1$ which, in turn, can be related to an adjustment in net benefits $\Delta T_{k,i,m}$ due to the adjusted inflow pattern. With this capability, estimates for all necessary variables are available for computation of the expected change in benefits at k due to the decision variable adjustment as per

$$E[\Delta\langle\hat{B}_{ki}\rangle] = \mathbf{W}\cdot[\Delta\langle\hat{\mathbf{B}}_{ki}\rangle^T\cdot\mathbf{P}_{ki} + \langle\mathbf{B}_{ki}\rangle^T\cdot \Delta\mathbf{P}_{ki} + \Delta\mathbf{T}_{ki}] \quad (17)$$

It is worthy of mention that there is no restriction in the gradient estimator that stipulates that only reservoirs connected in series can be analyzed. In other words, the analysis of reservoirs connected in parallel may be included (although the model cannot recognize joint operation in a short-term policy).

The method of solution of the complex basin problem described above necessitates knowledge of the activity in segment $k - 1$ prior to solution at k due to the need for $\Delta\mathrm{T}_{ki}$. Since $\Delta\mathrm{T}_{ki} = 0$ for the segment $k = 1$ (there are no segments downstream), the logical position to commence evaluation of the marginal analysis equations is at the bottom of the system. Successive computations may then simply work upstream.

Note that with the use of the analog a single run of the simulation model is sufficient to construct the transition matrices; subsequent analyses to determine the weighted sum of the marginal changes in net benefits for adjustments in the decision variables may be done entirely within the analog framework. Thus the analog provides a rational procedure for successive search of a complex multiobjective response surface. Information from the analog alone does not lead to the optimal solution but simply indicates the direction of a better solution set. Therefore the analog must be used in combination with a search technique to move in a sequence of steps, or simulation experiments, toward better or improved alternatives. The main contribution of this work then is to advance the computational capability for estimating response surface gradients so that existing search procedures can, in fact, be used.

SUMMARY

No single computer model can encompass all of the important issues implicit in the design of a water resources development plan, and so recourse in planning problems has been made to use of a series of computer models. The available models, however, have lacked a formal methodology for efficiently utilizing information gained from previous use of the models. One of the principal reasons for this has been the absence of an efficient means for estimating the gradient vector of the response surface. This paper has presented a procedure to compute the gradient through use of a probabilistic Markov model that modified the reservoir state transition probabilities as a function of the adjustment of the decision variables. The gradient estimators provided the necessary marginal analysis adjustments to develop the transformation surface in a multiple-objective environment.

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