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Selection of Δx and Δt Computational Steps for Four-Point Implicit Nonlinear Dynamic Routing Models

by

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Abstract. A major step in a successful application of unsteady flow models based on the numerical (four-point implicit, nonlinear finite-difference) solution of the complete one-dimensional Saint-Venant equations, is the selection of the magnitudes of the computational distance step (Δx) and time step (Δt) used in the numerical solution technique. A theoretical explanation is presented for the basis of the empirical selection criteria used rather successfully for a number of years; also, an enhanced time step selection criterion is presented. The suitability of the selection criteria is demonstrated using a numerical convergence testing technique for a wide spectrum of unsteady flow applications ranging from rapidly to slowly rising hydrographs in very flat to very steep sloping channels.

Introduction

Four-point implicit nonlinear finite-difference approximation equations of the complete Saint-Venant unsteady flow equations constitute the most extensively used basis of implicit dynamic routing models such as the NWS DAMBRK (Dam-Break), DWOPER (Dynamic Wave Operational) and FLDWAV (Flood Wave) (Fread, 1985, 1988). It is most important that appropriate computational distance (Δx) and time step (Δt) parameters be used in the application of these routing models. If the selected values are too small, the computations are inefficient, sometimes to the extent of making the application too expensive or time consuming and therefore infeasible; however, if the values are too large, the resulting truncation error (the difference between the true solution of the partial differential Saint-Venant equations and the approximate solution of the four-point implicit finite-difference approximations of the Saint-Venant equations) can cause significant errors in the computed discharges and corresponding water-surface elevations; and the errors may be so large as to make the computations totally unrealistic. Unrealistic solutions can cause the computer program

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to abort when computed elevations result in negative depths; also, unrealistic solutions can result in significant irregularities (spurious spikes) in the computed hydrographs.

Over several years of experience with the selection of Δx and Δt values for the NWS implicit dynamic routing models in numerous applications, the following empirical selection criteria evolved:

$$\Delta x \leq c\ T_r/20 \tag{1}$$

and, $$\Delta t \simeq T_r/20 \tag{2}$$

where T_r is the hydrograph's time of rise (time from the significant beginning of increased discharge to the peak of the discharge hydrograph), in hours; and c is the bulk wave speed (the celerity associated with an essential characteristic of the unsteady flow such as the peak or center of gravity of the hydrograph), in miles/hour; Δt is the computational time step size, in hours; and Δx is the computational distance step size, in miles. In most applications, the bulk wave speed is well approximated as a kinematic wave. Since c can vary along the waterway (channel, river, reservoir, estuary), Δx may not be constant. The kinematic wave celerity is approximated as:

$$c \simeq k'\ V \tag{3}$$

in which k' is the kinematic wave ratio having values ranging from $4/3 \leq k' \leq 5/3$ ($k' \simeq 3/2$ for most natural channels), and V is the flow velocity.

Herein, a theoretical explanation for the Δx and Δt empirical selection criteria is presented. The suitability of the selection criteria is demonstrated using numerical convergence testing for a wide spectrum of unsteady flow applications.

Theoretical Derivation of Δx and Δt Criteria

Theoretical wave damping (attenuation) and celerity (velocity) error (e)-diagrams were obtained previously by Fread (1974) using a Fourier technique to analyze linearized Saint-Venant equations. The e-diagrams showed convergence ratios (ratio of implicit finite-difference solution of linearized equations to their analytical solution) for wave damping and celerity plotted against D_L (wave discretization numbers) for a range of D_C (Courant numbers) values and (D_f) dimensionless friction numbers. Recently, a relationship between D_L and D_C was found for error values in the range of 0 to 5 percent; i.e.,

$$D_L \geq \eta\ D_C \tag{4}$$

where η is approximately 12 for $e \simeq 2$ percent, and $\eta \simeq 7$ for $e \simeq 5$ percent. The wave discretization number (D_L) is defined as:

$$D_L = L_w/\Delta x \tag{5}$$

where L_w is the wave length and Δx is the computational distance step. However,

$$L_w = c\ T \simeq c\ 3\ T_r \tag{6}$$

where c is the kinematic wave celerity, T is the wave period of the unsteady disturbance (wave), and T_r is the time of rise of the wave or hydrograph. Substituting Eq. (6) into Eq. (5) yields:

$$D_L = 3\ c\ T_r/\Delta x \tag{7}$$

The Courant number (D_C) is defined as:

$$D_C = c' \ \Delta t/\Delta x \tag{8}$$

where $c' = V + \sqrt{gD}$ (9)

in which c' is the dynamic wave celerity, V is flow velocity, g is the gravity acceleration constant, and D is the hydraulic depth of flow.

<u>Δx Selection Criteria</u>. Substituting Eq. (7) into Eq. (4) yields:

$$3 \ c \ T_r \ / \ \Delta x \geq \eta \ D_C \tag{10}$$

which can be rearranged to give:

$$\Delta x \leq \frac{c \ T_r}{\eta \ D_C/3} \tag{11}$$

If η is replaced with the conservative value of 12, i.e., a 2 percent level of truncation error is tolerated, and if $D_C \geq 5$, then

$$\Delta x \leq c \ T_r/20 \tag{12}$$

which is identical with the empirical formula for Δx selection, i.e, Eq. (1).

The Δx selection criterion, Eq. (12), is based on the linearized form of the Saint-Venant equations; however, the complete Saint-Venant equations used in the NWS implicit routing models are nonlinear. The nonlinear terms can interact with highly nonlinear data, e.g. channel properties, so as to require even smaller Δx computational distance steps (Fread, 1988) than specified by Eq. (12).

<u>Δt Selection Criterion</u>. In order to find an expression for the selection of Δt, Eq. (7) and Eq. (8) are substituted into Eq. (4). This gives:

$$3 \ c \ T_r \ / \ \Delta x \geq \eta \ c' \ \Delta t/\Delta x \tag{13}$$

which can be rearranged to give:

$$\Delta t \leq T_r/M \tag{14}$$

where $M = \eta \ c'/(3 \ c)$ (15)

Replacing η with the conservative value of 12 which allows a 2 percent level of truncation error, and substituting Eq. (3) and Eq. (9) into Eq. (15) yields:

$$M = 4 \ (V + \sqrt{gD})/(1.5V) \tag{16}$$

Now, the Manning equation is used for V, i.e.,

$$V = \mu' \ D^{2/3} \ S_o^{1/2}/n \tag{17}$$

in which μ' is 1.49 (1.0 if SI units), D is hydraulic depth, S_o is bottom slope, and n is the Manning roughness coefficient. Substituting this in Eq. (16) gives:

$$M = 2.67\left[1 + \tilde{\mu} \ n^{0.9}/(q^{0.1} \ S_o^{0.45})\right] \tag{18}$$

in which $\tilde{\mu}$ is 3.97 (US units) and 3.13 (SI units), and q is the average unit width discharge along the routing reach. Using typical values for S_o, n, and q provides a

range of M values generally not exceeding $6 \leq M \leq 30$. Thus, Eq. (14) with an M value of 20 is the same as Eq. (2). Unlike Eq. (2), Δt is variable in Eq. (18).

Convergence Testing to Validate Δx and Δt Selection Criteria

Numerical convergence testing is a technique wherein sensitivity tests are performed for a given problem to see if a sequence of unsteady numerical solutions with increasingly refined computational distance (Δx) and time (Δt) steps approach a fixed value, i.e., the numerical solution has converged if further refinement of Δx and Δt produces insignificant change in the solution.

Convergence testing is applied to three cases spanning a wide spectrum of unsteady flow applications ranging from rapidly to slowly rising hydrographs in very flat to steep sloping channels. In each case, the channel is 100 ft (30 m) wide with a constant Manning n of 0.045, an initial flow of 2000 cfs (56.6 cms), and a single-peaked inflow hydrograph with Q_{peak} = 20000 cfs (566.3 cms). Case (1) has a flat channel slope of 0.0002 and an inflow hydrograph with T_r = 1.0 hr; the routing reach is 20 miles long; convergence testing is at mile 10. Case (2) is identical to case (1) except the channel slope is steep (0.01). Case (3) has a very flat channel slope of 0.0001 and an inflow hydrograph with T_r = 72 hr; the routing reach is 100 miles; convergence testing is at mile 50.

Case (1) (the flat channel and rapidly rising hydrograph) convergence testing results are shown in Fig. 1a for Δt of 0.05 hr and Δx distance steps of 5, 2.5, 2.0, 1.0, 0.5, and 0.25 miles. Erroneous leading waves appear for all distance steps, $\Delta x \geq 1.0$ mile, and the hydrograph peak ceases to vary by less than 0.1 percent for the 0.5 mile distance step. Eq. (12), using a wave speed (c) of 7.0 mi/hr, gives a Δx value of 0.36 mile which is near that (0.5 mile) obtained from the convergence testing. As indicated in Fig. 1b, convergence seems to be is reached with the time step of about 0.05 hr. This is in agreement with Eq. (14) which also yields a value of 0.05 hr using a computed value of 21 for M from Eq. (18).

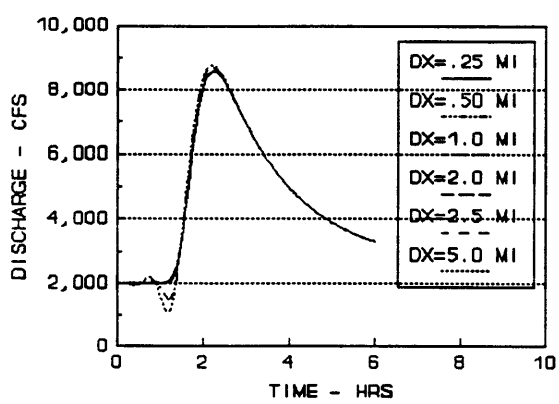

Fig. 1a Convergence testing for Δx (Case 1)

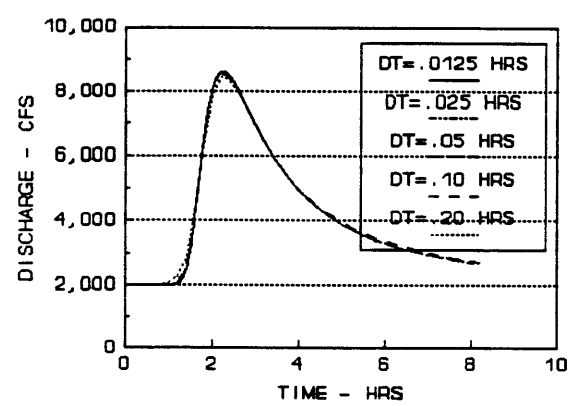

Fig. 1b Convergence testing for Δt (Case 1)

Case (2) (the steep channel and rapidly rising hydrograph) convergence testing results are shown in Fig. 2 for a Δt of 0.10 hr. Convergence appears to be reached with a distance step $\Delta x \leq 2.0$ miles compared with a value of 1.67 miles provided by Eq. (12) using a wave celerity (c) of 33.3 mi/hr.

Case (3) (the very flat channel and slow rising hydrograph) convergence testing results are shown in Fig. 3 for a Δt of 3 hr. Convergence appears to be attained for $\Delta x \leq 25$ miles compared to a computed value of 15 miles provided by Eq. (12) using a wave celerity of 4.2 mi/hr.

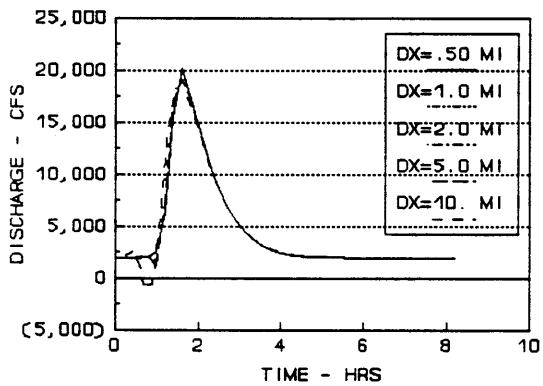

Fig. 2 Convergence Testing for Δx (Case 2)

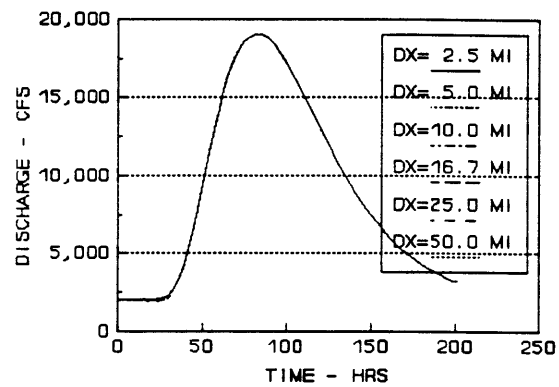

Fig. 3 Convergence Testing for Δx (Case 3)

Conclusions

Critical to applications of Saint-Venant based implicit dynamic routing models is the selection of the computational distance steps (Δx) and time steps (Δt). A theoretical derivation is given for the Δx and Δt selection criteria. These criteria not only explain the utility of the previous empirical formulae in producing acceptable computational results, but are also capable of yielding appropriate Δx and Δt values for routing applications significantly differing from past experience. The validity of the selection criteria is demonstrated through convergence testing.

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