A MATHEMATICAL MODEL FOR QPF FOR FLOOD FORECASTING PURPOSES

By
P. N. SEN

Abstract

A mathematical model for Quantitative Precipitation Forecasting has been developed on the basis of physical and dynamical laws. The surface and upper air meteorological observations have been used as inputs in the model. The output is the rate of precipitation from which the amount of precipitation can be computed on time integration. The model can be used operationally for rainfall forecasting.

INTRODUCTION: An accurate prediction of the amount of precipitation is of immense importance to a wide range of human activity. It is particularly significant for forecasting of floods which may in turn help in saving the lives of thousands of people and damage to property every year. Attempts have been made in several countries to forecast the amount of precipitation in different catchment areas with different lead time (the difference between the time of the occurrence of the forecasted phenomenon and the time when the forecast is issued) using various techniques but have achieved only limited success. A very good account on the status of Quantitative Precipitation Forecasting (QPF) models used in various countries for operational purposes has been described by Bellocq (1980). The recent advances in QPF research and possible future directions toward achieving improved use of QPF information in hydrological forecasting have been discussed at length by Georgakakos and Hudlow (1984).

The main rain producing system in the Indian Subcontinent is the South West Monsoon or Asiatic Summer Monsoon (June to September) which is typical of

this part of the world. Some parts or other in India come under the grip of flood due to heavy rainfall in the catchment areas during the South West Monsoon period, every year, causing death to people and animals and damage to standing crops and property. India Meteorological Department has been issuing QPF for various river catchments in India from the Flood Meteorological Offices (FMO) located at various locations using different techniques (viz. Synoptic, Statistical, Synoptic-statistical) for more than a decade. But so far no mathematical model has been developed for QPF in India for operational purposes.

In this paper an attempt has been made to develop a mathematical model for QPF, based on physical and dynamical laws, which could be used for operational purposes. The inputs are the surface and upper air meteorological observations and the output is the rate of precipitation from which the amount of precipitation can be computed readily on time integration. This forecast amount of rainfall can again be used as an input in hydrological model for flood forecasting purposes, if desired.

FORMULATION AND DESCRIPTION OF THE MODEL: The atmospheric model is designed to recognize the presence of water either as vapor or liquid. The concentration of water vapor may be represented by the Specific Humidity, q, defined as the ratio of the mass of water vapor to the mass of moist air.

The present mathematical model is based on the hypothesis that the specific humidity in an air column is conserved. In other words, it is based on the principle of conservation of specific humidity. In mathematical notation it can be expressed as:

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + \frac{u \partial q}{\partial x} + \frac{v \partial q}{\partial y} + \frac{\omega \partial q}{\partial p} = 0 \tag{1}$$

where $u = \frac{dx}{dt}$, $v = \frac{dy}{dt}$, and $\omega = \frac{dp}{dt}$.

The precipitable water vapor in a column of air is the total mass, R_F, of water vapor per unit area in the column. Symbolically (Haltiner and Martin, 1957)

$$R_F = \int_{z_s}^{z_t} \rho_v \, \delta z \tag{2}$$

where ρ_v is the density of water vapor and z_s and z_t the elevations of the bottom and the top of the air column respectively.

The relation (2) can be written in the isobaric coordinate as

$$R_F = -\int_{p_s}^{p_t} \frac{\rho_v}{g\rho} \, \delta p \tag{3}$$

where ρ is the density of air and p_s and p_t the pressure at the bottom and at the top of the air column respectively and g, the acceleration due to gravity.

The equation (3) can be written as

$$R_F = -\frac{1}{g} \int_{p_s}^{p_t} q \, \delta p \tag{4}$$

where $q = \rho_v / \rho$ = Specific Humidity

Since q is a nondimensional quantity, the variable R_F has the dimensions of mass per unit area. R_F may be converted to the parameter "Precipitable Water" by dividing R_F by the density of water.

We can now derive an expression for the Rate of Precipitation, $\frac{\partial R}{\partial t}$, from equation (4) in the following way

$$\frac{\partial R}{\partial t} = E \frac{\partial R_F}{\partial t} \tag{5}$$

where $E(0 \leq E \leq 1)$ is a multiplication factor. The multiplication factor has been used because R_F is the amount of precipitable water vapor not the actual amount of precipitation. $R_F = R$, if and only if all the available moisture condenses and falls as precipitation. But in the actual atmosphere it has been found that only a part of the precipitable water vapor gets converted into precipitation. Thus E is a measure of the proportion of the available moisture which precipitates and may be termed as Precipitation Efficiency. This is a key parameter which takes care of moisture loss due to evaporation and other aspects of cloud microphysics.

Combining equations (4) and (5) we get

$$g \frac{\partial R}{\partial t} = gE\frac{\partial R_F}{\partial t} = -E\frac{\partial}{\partial t} \left\{ \int_{p_s}^{p_t} q \, \delta p \right\} = -E\left\{ \int_{p_s}^{p_t} \frac{\partial q}{\partial t} \, \delta p \right\} \tag{6}$$

It has been assumed in the above that the variation of p_t and p_s with time counterbalance each other.

Using equation (1) we get from equation (6)

$$g \frac{\partial R}{\partial t} = E \int_{p_s}^{p_t} \left(\frac{u\partial q}{\partial x} + \frac{v\partial q}{\partial y}\right) \delta p + E \int_{p_s}^{p_t} \omega \frac{\partial q}{\partial p} \delta p$$

$$\text{or, } g \frac{\partial R}{\partial t} = E \sum_{i=1}^{n} (p_{i+1} - p_i) \left< u \frac{\partial q}{\partial x} + \frac{v\partial q}{\partial y} \right>_{i,i+1} + E \sum_{i=1}^{n} (q_{i+1} - q_i) \langle \omega \rangle_{i,i+1} \tag{7}$$

where i=1, corresponds to the surface pressure p_s and i=n to the pressure surface p_t and $\langle\ \rangle_{i,i+1}$ represents average value of the layer bounded by the surfaces i and i+1.

The equation (7) can also be written as

$$g \frac{\partial R}{\partial t} = E \sum_{i=1}^{n} (\Delta p)_{i,i+1} \langle \vec{V} \cdot \nabla q \rangle_{i,i+1} + E \sum_{i=1}^{n} (\Delta q)_{i,i+1} \langle \omega \rangle_{i,i+1} \tag{8}$$

where $(\Delta p)_{i,i+1} = p_{i+1} - p_i$; $(\Delta q)_{i,i+1} = q_{i+1} - q_i$

The first term on the right-hand side of the equation (8) actually represents the horizontal advection of moisture and may be termed as the Advection Term and the second term represents the effect of the vertical velocity on moisture and may be termed as the Vertical Velocity Term. Thus equation (8) shows that the Velocity Field especially the vertical velocity coupled with information on humidity, is related to the rate of precipitation in a large area.

Precipitation is the end product of the physical processes taking place in the atmosphere. The occurrence of precipitation is strongly controlled by the motion of the cloud air. In other words, rainfall is always associated

with clouds and the moisture advection normally takes place at the boundary layer which almost coincides with the base of the clouds. It would be quite reasonable if 850 mb level is chosen as the top of the boundary layer. In that case the summation in the first term on the right-hand side of equation (8) need not be performed up to the top of the air column; the summation up to the top of the boundary layer would be sufficient. Thus the equation (8) may be modified as

$$g \frac{\partial R}{\partial t} \simeq E\ (\Delta p)_{p_{s},\ 850} \langle \vec{V} \cdot \nabla q \rangle_{p_{s},\ 850} + E \sum_{i=1}^{n} (\Delta q)_{i,i+1} \langle \omega \rangle_{i,i+1} \qquad (9)$$

Since the parameter q, the specific humidity, is not directly measured it is desirable that it be expressed in terms of some parameters, those are either measured at the observational sites or at least reported in the synoptic or upper air observations.

By definition, the specific humidity is given by

$$q = \frac{\rho_v}{\rho} = \frac{\rho_v}{\rho_d + \rho_v}, \text{ where } \rho_d = \text{density of the dry air}$$

$$q = \frac{0.622\ e_s(T_d)}{p - 0.378 e_s(T_d)} \qquad (10)$$

where $e_s(T_d)$ is the saturation vapor pressure over a plane surface of pure water; T_d, the dew point temperature. Since $e_s(T_d) \ll p$, the equation (10) can be written as

$$q \simeq \frac{0.622}{p} e_s(T_d) \qquad (11)$$

The saturation vapor pressure is a non-linear Convex function of temperature. A convenient formulation for determining the saturation vapor pressure $e_s(T_d)$ in terms of T_d may be used for this purpose. The polynomial relation suggested by Lowe and Ficks (1974) has been found to provide an excellent fit with the observed ones in the range -50°C to +50°C (Pruppacher and Klett, 1980).

The formulation reads as

$$e_s(T_d) = \sum_{n=0}^{6} a_n T_d^n, \text{ with } T_d \text{ in } {}^0\text{C and } e_s \text{ in mb} \qquad (12)$$

where $a_0 = 6.107799961$, $a_1 = 4.436518521 \times 10^{-1}$, $a_2 = 1.428945805 \times 10^{-2}$

$a_3 = 2.650648471 \times 10^{-4}$, $a_4 = 3.031240396 \times 10^{-6}$, $a_5 = 2.034080948 \times 10^{-8}$

$a_6 = 6.136820929 \times 10^{-11}$

Combining equations (11) and (12) we get

$$q = \frac{0.622}{p} \sum_{n=0}^{6} a_n T_d^n \qquad (13)$$

where T_d is expressed in °C and p in mb and q is a nondimensional quantity.

Using equations (9) and (13) the rate of precipitation may be computed provided the value of E is available. Normally the rate of precipitation is expressed in $Kgm^{-2}\ sec^{-1}$. But since 1 $Kg\ m^{-1}$ of liquid water is equivalent to the depth of 1 mm of rainfall, $\frac{\partial R}{\partial t}$ can be expressed directly as mm sec^{-1}. The

vertical p- velocity ω may be computed using either the equation of continuity in isobaric coordinates or the diagnostic ω equation. The second method is more accurate.

The method suggested by Sulakvelidze (1969) and applied by Georgakakos and Bras (1984a and b) for vertical velocity can also be used. The expression for the vertical velocity w used by them is

$$w = a \sqrt{c_p \Delta T} \qquad (14)$$

where a is a constant parameter, $\Delta T = |T_m - T_s|$, T_m is the parcel temperature [°K] at a certain level p [mb] assuming pseudoadiabatic ascent and T_s is the corresponding ambient air temperature [°K]. The square of the quantity 'a' is analogous to the ratio of the Kinetic to the Thermal Energy per unit mass of ascending air, at the level p. Therefore, 'a' is a nondimensional quantity. When the parcel temperature is more than the environmental temperature there would be upward vertical motion and opposite is the case when the parcel is cooler than the environment. Georgakakos and Bras (1984a and b) have found out that a value of 0.002 for 'a' gave a good fit. From the equation (14) we can compute the vertical p- velocity ω using the following relation (Holton 1979)

$$\omega \simeq - g \rho w \qquad (15)$$

ESTIMATION OF PRECIPITATION EFFICIENCY: The quantity Precipitation Efficiency, E, is a complex, elusive factor to quantitatively determine. It is not theoretically spatially constant either. Rhea (1978) used

precipitation efficiency (which is slightly different from the precipitation efficiency defined here) in his Orographic Precipitation Model and reported that many computations of precipitation rates for hydrometeorological purposes routinely set E = 1 (i.e., they equate condensation supply rate to precipitation rate). A large number of test cases were run by Rhea using a variety of E values and it was found that on 50% occasions a value of 0.25 for E and at least 70% occasions a value of 0.21 or greater gave a good fit with the observed values. Moreover, it has been estimated that only about 30% of the moisture falls out as precipitation (Haltiner and Martin, 1957). Thus it would be reasonable if a value of 0.20 for E is chosen considering the loss due to evaporation of the droplets of precipitation. But it is recommended that the value of E be estimated for each individual station with a long series of data with varieties of rain storms. However, it is to be remembered that extreme parameter sensitivity is not desirable considering the crudeness of the input data as well as the precipitation measurement.

DISCUSSION: The model described in this paper is very simple and can be used for operational purposes. The model is based on the hypothesis that the specific humidity of the atmospheric column under consideration is conserved. This assumption is valid as long as there is no change in phase, which means once the process of condensation starts the specific humidity may not remain constant. Since in this model we are calculating the rate of change of precipitable water vapor in the atmospheric column and the rate of precipitation is computed from it through a factor called precipitation efficiency, the above assumption may be considered valid. The same hypothesis has been taken into consideration in the Limited-Area, Fine-Mesh Model (LFM) in the National Weather Service (NWS) of the United States of America (Gerrity

1977; Newell and Deaven 1981). Several workers in U.S.S.R. attempted to develop a method to forecast cloudiness and precipitation on the basis of the same hypothesis (for complete list see Matveev, 1967a).

This model may be reasonably successful for predicting precipitation associated with large scale (Synoptic scale) disturbances and may not be successful to that extent for the prediction of orographic and convective precipitation. The orographic vertical motion is frequently an order of magnitude larger than that associated with large scale vertical velocity, while the vertical motion associated with mesoscale convective system may be still higher. Both orographic and convective element vertical motions are small scale phenomena, implying action on a given air parcel for only a short time whereas the large scale vertical motion field slowly displaces a given parcel for an extended period. Thus each may have a considerable influence on the total precipitation process.

Matveev (1968b) has reported an interesting phenomenon associated with Cumulonimbus (Cb) cloud. He has reported from the data of 26 cases in the Kiev region in Soviet Union that the precipitation exceeded the cloud precipitable water content on the average by a factor of 8.8 (with variations between 1.8 and 16.9). The water reserves of a Cb cloud are replenished every 7-12 minutes. The above data show that the amount of precipitation from the Cb cloud systems during their time of existence exceeds by about one order of magnitude of their water content at any given moment. This means that the water is completely renewed many times during the cloud's existence. The above observation confirms the fact that the forecasting of the amount of rainfall associated with the convective cloud is a very difficult proposition if not impossible. As such QPF itself, for any type of precipitation, is a very difficult task.

Moreover, unlike other meteorological parameters rainfall is a highly variable quantity. Precipitation amounts are seldom representative, and a few rain gauges do not constitute an adequate sample of a large area for quantitative purposes. Moreover, there are several methods for determining the average depth of precipitation and the amount computed by one method differs considerably from that computed by another method.

Acknowledgement

The author would like to thank Mr. S.K. Shosh, Director, Regional Meteorological Center, New Delhi, India; Dr. Michael D. Hudlow, Chief, Hydrologic Research Laboratory, National Oceanic and Atmospheric Administration (NOAA), U.S.A.; Dr. Konstantine P. Georgakakos and Dr. Witold F. Krajewski, also of Hydrologic Research Laboratory, NOAA, for helpful discussions and critical comments. Thanks are also due to Mrs. Mildred H. Larson for efficient typing of the material.

REFERENCES

1. Bellocq, A. (1980): Operational Models of Quantitative Precipitation Forecasts for Hydrological Purposes and Possibilities of an Intercomparison; World Meteorological Organization, Geneva, Switzerland.

2. Georgakakos, K.P. and R. L. Bras (1984): a) A Hydrologically Useful Station Precipitation Model. 1. Formulation--Water Resources Research, vol. 20, No. 11, pp 1585-1596; b) A Hydrologically Useful Station Precipitation Model. 2. Case studies - Water Resources Research, Vol. 20, No. 11, pp. 1597-1610.

3. Georgakakos, K.P. and M. D. Hudlow (1984): Quantitative Precipitation Forecast Techniques for use in Hydrologic Forecasting. Bull. Amer. Met. Soc, Vol. 65, No. 11, pp. 1186-1200.

4. Gerrity, J. P. (1977): The LFM Model - 1976, A Documentation, NOAA Technical Memorandum NWS NMC 60.

5. Haltiner, G.J. and F.L. Martin (1957): Dynamical and Physical Meteorology, McGraw Hill Book Company, Inc., New York p 55.

6. Holton, J. R. (1979): An Introduction to Dynamic Meteorology, Second Edition, Academic Press, New York, p 72.

7. Lowe, P.R. and J. M. Ficke (1974): Techn. Paper No. 4-74, Environmental Prediction Res. Facility, Naval Post Grad School, Monterey, California.

8. Matveev, L. T. (1967): Fundamentals of General Meteorology - Physics of the Atmosphere, Translated from Russian by Israel Program for Scientific Translation, Jerusalem a) p. 466; b) p. 490.

9. Newell, J. E. and D. G. Deavan (1981): The LFM-II Model- 1980, NOAA Technical Memorandum NWS NMC 66.

10. Pruppacher, H. R. and J. D. Klett (1980): Microphysics of Clouds and Precipitation, D. Reidel Publishing Company, Dordrecht, Holland, p. 625.

11. Rhea, J. O. (1978): Orographic Precipitation Model for Hydrometeorological Use, Ph.D Dissertation, Colorado State University, Atmospheric Science Paper No. 287.

12. Sulakvelidze, G. K. (1969): Rainstorms and Hail, Translated from Russian by the Israel Program for Scientific Translation, Jerusalem, p. 19.