Computational Extensions to Implicit Routing Models+

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ABSTRACT: Dynamic flood routing models based on an implicit nonlinear finite difference solution of the Saint-Venant equations have been extended computationally to allow a larger range of practical applications and greater reliability and ease of use. This paper presents a brief description of the following computational developments: (a) a computationally efficient algorithm for treating transient flows in channel networks, (b) convenient selection of various internal boundary conditions to simulate rapidly varied flows at dams, bridges, etc., (c) an algorithm for simulating the effects of levee overtopping, (d) an algorithm for stable computation of mixed (subcritical-supercritical) unsteady flow, (e) an automatic computational stability enhancement via a temporary reduction in time step size or increase in the θ weighting factor of the finite difference approximating equations, and (f) an algorithm to create additional cross sections via linear interpolation between two adjacent sections.

Introduction

During the last few years, operational dynamic flood routing models based on a weighted, four-point, implicit, nonlinear finite difference solution of the one-dimensional unsteady flow (Saint-Venant) equations have been increasingly used to analyze dam-break flooding and other transient flows in mild-sloping river systems affected by tides, backwater and man-made structures (dams, bridges-embankments, levees, flow bypasses, etc.). This paper presents a brief description of several practical concepts and algorithms which have been developed to significantly extend the range, reliability, and ease of application of two operational implicit flood routing models, DWOPER and DAMBRK previously described by the author (2,3). Although the computational extensions presented were developed for particular models, most are generally applicable to dynamic routing models of the implicit (linear or nonlinear) or explicit type.

Computational Extensions

Networks.—A network of channels presents complications in achieving computational efficiency when using the implicit formulation. Equations representing the conservation of mass and momentum at the confluence of two channels produce a Jacobian matrix in the Newton-Raphson method with elements which are not contained within the narrow band along the main-

+Published in "Frontiers in Hydraulic Engr", H.T. Shen, Editor, ASCE Hydraulics Specialty Conf., Aug. 9-12, 1983, MIT, Cambridge, Mass., pp. 343-348.
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diagonal of the matrix. The column location of the elements within the Jacobian depends on the sequence numbers of the adjacent cross sections at the confluence. The generation of such "off-diagonal" elements produces a "sparse" matrix containing relatively few non-zero elements. Unless special matrix solution techniques are used for the sparse matrix, the computation time required to solve the matrix by conventional matrix solution techniques is so great as to make the implicit method infeasible. The same situation also occurs for the linearized implicit methods which must also solve a system of linear equations similar to the Jacobian. Two algorithms have been used by the author for an efficient computational treatment of channel networks.

The first, called the "relaxation" algorithm, is restricted to a dendritic (tree-type) network of channels in which the main channel has any number of tributary channels joining with it. Sometimes, dendritic systems with second-order tributaries (tributaries of tributaries) can be accommodated in the relaxation technique by reordering the dendritic system, i.e., selecting another branch of the system as the main channel. In the relaxation algorithm, no sparse matrix is generated; the Jacobian is always banded as it is for a single channel reach. This algorithm has been described previously (1).

The second, called the "network" algorithm, can be used on almost any natural system of channels (dendritic systems having any order of tributaries; bifurcating channels such as those associated with islands, deltas, flow bypasses between parallel channels; and tributaries joining bifurcated channels). The network algorithm produces a sparse matrix which is solved by a special matrix technique which treats only non-zero elements. The relaxation algorithm is somewhat more efficient than the network algorithm, but the latter is more versatile.

The network algorithm is based on the treatment of the channel junctions (confluences, bifurcations) as internal boundary conditions using the following three equations:

$$Q_i^{j+1} + Q_{i'}^{j+1} - Q_{i'+1}^{j+1} - \Delta s/\Delta t^j = 0 \tag{1}$$

$$2g\left(h_i^{j+1} - h_{i'+1}^{j+1}\right) + \left(Q^2/A^2\right)_i^{j+1} - T\left(Q^2/A^2\right)_{i'+1}^{j+1} = 0 \tag{2}$$

$$2g\left(h_{i'}^{j+1} - h_{i'+1}^{j+1}\right) + \left(Q^2/A^2\right)_{i'}^{j+1} - T\left(Q^2/A^2\right)_{i'+1}^{j+1} = 0 \tag{3}$$

where: $$\frac{\Delta s}{\Delta t} = \Delta x_i/(6\Delta t^j)\,\tilde{B}\left(h_i^{j+1} + h_{i'}^{j+1} + h_{i'+1}^{j+1} - h_i^j - h_{i'}^j - h_{i'+1}^j\right) \tag{4}$$

$$i' = i + m + 1 \tag{5}$$

$$\tilde{B} = B_i^j + B_{i'+1}^j + B_{i'}^j \cos \omega_t \tag{6}$$

$$T = 1 + C_m + C_f \tag{7}$$

$$C_m = \left(0.1 + 0.83\, Q_{i'}^j/Q_{i'+1}^j\right)\left(\omega_t/90\right)^{\mu} \tag{8}$$

$$C_f = 2g \; \Delta x_i \; \bar{n}^2/[2.21 \; (\bar{D}^{4/3})^j \tag{9}$$

in which $\bar{D}$ is the average depth in the junction, $\bar{n}$ is the Manning n for the junction, ω_t is the acute angle between the upstream reach and the branch, μ is an exponent assumed to be unity, and m is the total number of Δx reaches located upstream (downstream) along the branching channel. The parameters C_f and C_m are related to friction effects and to the head loss due to mixing as reported by Lin and Soong (4), respectively. The superscript (j) and the subscript (i) represent respectively the time line and cross section location in the x-t computational plane.

Computational efficiency is achieved by minimizing the number of off-diagonal elements in the Jacobian and by minimizing the creation of new off-diagonal elements during the elimination phase of the matrix solution. Also, the way in which the cross sections are assigned sequential numbers within the channel network is important in effecting the desired minimization. The numbering scheme is as follows: numbers run consecutively in the downstream direction until a dendritic-type junction is reached; then the most upstream section of the dendritic branch is given the next consecutive number and the numbers increase in the downstream direction along this branch until another junction is reached; then the most upstream section of that dendritic branch is numbered and the numbers increase in the downstream direction along that branch until a new junction is reached; this is repeated until all sections have been numbered, including the first cross section of the branch of the very first dendritic-type junction; then the numbers continue to increase along the downstream branch of this junction. Bifurcations are numbered in a similar manner.

Computational efficiency is achieved also by use of a specially developed matrix solution technique of the Gauss elimination type which operates on only non-zero elements in the matrix through use of a specified code number for each cross section in the network of channels. The specified code number is as follows: (1) regular cross section, (2) upstream boundary, (3) downstream boundary, (4) dendritic-type junction, (5) dendritic-type junction emanating from a bifurcated channel branch, (6) upstream junction of a bifurcation around an island, (7) downstream junction of a bifurcation around an island, (8) bifurcation-type junction emanating from another bifurcated channel and joining with a third bifurcated channel, and (9) bifurcation-type junction emanating from a bifurcated channel and joining into the other branch of the bifurcation.

The Jacobian is a 2N x 2N matrix where N is the total number of cross sections. The number of operations (addition, subtraction, multiplication, division) required to solve the matrix is approximately (102 + 46J)N, where J is the total number of junctions. This is compared to (95N-48) operations for the relaxation algorithm, (38N-19) for a single channel and $(5N^3+8N^2+5N)$ for a standard Gauss elimination method.

Internal Boundaries.—There may be locations such as a dam, bridge, or waterfall (short rapids) along a waterway where the Saint-Venant equations are not applicable. At these locations, the flow is rapidly varied rather than gradually varied as necessary for the use of the Saint-Venant equations. Empirical water elevation-discharge relations such as weir-flow can be utilized for simulating rapidly varying flow.

Unsteady flows are routed along the waterway including points of rapidly varying flow by utilizing internal boundaries. At internal boundaries, cross sections are specified for the upstream and downstream extremities of the section of waterway where rapidly varying flow occurs. The Δx reach length between the two cross sections can be any appropriate value from zero to the actual measured distance. Since, as with any other Δx reach, two equations (the Saint-Venant equations) are required, the internal boundary Δx reach requires two equations. The first of the required equations represents the conservation of mass with negligible time-dependent storage, i.e.,

$$Q_i^{j+1} - Q_{i+1}^{j+1} = 0 \tag{10}$$

The second of the required equations can be any appropriate empirical rapidly varied flow relation between discharge (Q_i) and the upstream and downstream water surface elevations, e.g., the flow through a dam spillway and/or breach, a bridge, a critical flow section, or the overtopping flow of a bridge embankment.

Levee Effects.--Flows which overtop a levee located along either side or both sides of a channel may be simulated since any number of Δx reaches may bypass flow via a broadcrested wier-flow equation to another channel which represents the floodplain (beyond the levee). The floodplain channel may either directly connect back into the waterway at some downstream location, or it may be disconnected as in the case of the floodplain within a ringed levee where the flow is ponded with no exit. The hydraulic connection may be either a natural confluence or a flap-gated gravity drainage pipe. The flow in the floodplain can affect the overtopping levee flows via a submergence correction factor $K_{\ell e}$ similar to that used at internal boundaries of dams. The flow may also pass from the waterway to the floodplain through a time-dependent crevasse (breach) in the levee.

The overtopping and/or breach flow is routed through the floodplain which is considered as a tributary of the waterway along which the levee is located. The tributary (floodplain) channel must have a fictitious low-flow channel in which a small steady flow occurs at all times before the lateral inflow from the overtopped (breached) levee enters. The low flow, specified via the upstream boundary condition for the tributary, is necessary so that the Saint-Venant equations can be continuously solved during the simulation; however, at the hydraulic connection with the channel, the fictitious low flow is not added to the channel flow nor is it included in the flow that ponds within a ringed levee.

Depending on the relative elevations in the channel and floodplain (tributary), the overtopping levee flow can reverse its direction and flow from the floodplain back into the channel. Each Δx_i reach for the channel has a corresponding Δx_m reach along the floodplain channel. Each Δx_i reach has a submergence correction factor ($K_{\ell e}$), a broadcrested weir flow coefficient ($C_{\ell e}$), and a mean elevation ($\bar{h}_{\ell e}$) of the top of the levee. The effect of the levee flow is achieved by considering it to be lateral inflow or outflow (q) in the Saint-Venant equations. When routing the flow in the channel, if the flow overtops the levee and enters the floodplain, it is considered as bulk lateral outflow. When routing in the floodplain, the levee overtopping flow is considered as lateral inflow. The overtopping flow is computed as follows:

$$q_{\ell e_i} = S_g \, C_{\ell e_i} \, K_{\ell e_i} \left(\hat{h} - h_{\ell e_i}\right)^{3/2} \qquad \ldots \; \hat{h} > h_{\ell e_i} \tag{11}$$

where: $$S_g = (\tilde{h} - \hat{h})/|\tilde{h} - \hat{h}| \tag{12}$$

$$K_{\ell e_i} = 1. - 27.8\ (\gamma - 0.67)^3 \qquad \ldots \; \gamma > 0.67 \tag{13}$$

$$\gamma = \left(\tilde{h} - h_{\ell e_i}\right)/\left(\hat{h} - h_{\ell e_i}\right) \tag{14}$$

in which S_g determines the appropriate sign (- is outflow, + is inflow), $\hat{h}$ is the average water elevation along the Δx_i reach, $\tilde{h}$ is the average water elevation along the same Δx reach of the floodplain. Of course, the lateral flow may be zero when the water elevations do not overtop the levee or when the elevations are exactly the same. The overtopping levee flow is assumed to enter perpendicular to the direction of flow in the floodplain. Thus, it does not affect the conservation of momentum except when it is considered as bulk lateral outflow.

Mixed Flow.--When the flow changes with either time or distance along the routing reach from supercritical to subcritical or, conversely, the flow is described as "mixed". During each time step, subreaches are delineated where supercritical or subcritical flow exists by computing the Froude number at each cross section and grouping consecutive Δx_i reaches into either subcritical or supercritical subreaches. Then, the Saint-Venant equations are applied and solutions obtained for each subreach, commencing with the most upstream subreach and progressing downstream until each subreach has been solved. Appropriate external boundary equations are used for each subreach.

Where the flow changes from subcritical to supercritical, the downstream boundary for the subcritical subreach is the critical flow equation. The two upstream boundary equations for the supercritical subreach are the computed flow at the downstream boundary of the subcritical subreach and the computed critical depth. The supercritical subreach does not require a downstream boundary equation.

When the flow changes from supercritical to subcritical, the upstream boundary equation for the subcritical subreach is the computed flow at the downstream section of the supercritical subreach. The downstream boundary for the subcritical subreach would be the critical flow equation if another supercritical subreach exists below the subcritical subreach; if not, then an appropriate downstream boundary condition for subcritical flow would be used. The depth of flow at the first section of the subcritical subreach is determined by the downstream boundary condition and the Saint-Venant equations applied to the subcritical subreach. A hydraulic jump occurs between the last section of the supercritical subreach and the first section of subcritical subreach, although an equation for such is not directly used. To account for the possible movement of the hydraulic jump, the following procedure is utilized before advancing to the next time step: (a) the water elevation at the first section of the subcritical subreach is extrapolated to several upstream cross sections near the downstream end of the supercritical subreach; (b) the sequent depths (water elevations) of the same sections in the supercritical reach are computed; and (c) the sequent elevations are compared with the extrapolated elevations, and the first section of the subcritical subreach is determined as that section nearest the intersection of the two elevations.

Enhancement of Basic Computational Algorithm.--An automatic procedure which increases the robust nature of the four-point, nonlinear implicit finite difference algorithm is quite useful when treating rapidly rising hydrographs in channels where the cross sections have large variations in the vertical and/or along the x-axis. This situation may cause computational problems which are manifested by non-convergence in the Newton-Raphson iteration. When this occurs, the following procedure is implemented.

The current time step (Δt) is halved and the computations are repeated. If non-convergence persists, Δt is again halved and the computations repeated. This continues until a successful solution is obtained or the time step has been reduced to 1/16 of the original size. If a successful solution is obtained, the computational process proceeds to the next time level using the original Δt. If the solution using $\Delta t/16$ is unsuccessful, the Θ weighting factor is increased by 0.1 and a time step of $\Delta t/2$ is used. Upon achieving a successful solution, Θ and the time step are restored to their original values. Unsuccessful solutions are treated by increasing Θ and repeating the computation until $\Theta = 1.0$ whereupon the automatic procedure terminates and the solution with $\Theta = 1.$ and $\Delta t/2$ is used to advance the solution forward in time but using the original Θ and Δt values. Often, computational problems can be overcome via one or two time step reductions.

Linearly Interpolated Cross Sections.--An algorithm to generate additional cross sections between any two adjacent specified cross sections has been found to be quite useful. The properties of the additional sections are linearly interpolated from those of the end sections. This facilitates adherence to appropriate Δx reach lengths to insure computational accuracy and the use of smaller Δx distances in expanding or contracting reaches. Both active and off-channel storage widths are generated via the interpolation procedure.

Appendix - References

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