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A "UNIFIED" COEFFICIENT ROUTING MODEL<sup>1</sup>

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## INTRODUCTION

Many different simplified routing models have been developed throughout the last 50 years. Some of the more popular models are: the Reservoir Routing Model (Puls, 1928; Goodrich, 1930); the Muskingum Model (McCarthy, 1938); the Lag and K Model (Linsley et al. 1949); the Kinematic Wave Model (Lighthill and Whitham, 1955); the Kalinin-Miljukov Model (1958); the SSARR Model (Rockwood, 1958); the Muskingum-Cunge Model (Cunge, 1969); and the SWMM Model (Huber et al., 1975). Each has been widely used at different places and times. Currently, the Muskingum-Cunge Model is receiving much attention (Miller and Cunge, 1975; Ponce and Yevjevich, 1978; Koussis, 1978; and Weinmann and Laurenson, 1979) because: (1) it possesses an inherent potential for greater accuracy since it is a diffusion-type model rather than a kinematic-type model, the category in which the other models belong; and (2) its coefficients can be evaluated from the channel's hydraulic properties.

Herein, a routing model is developed which encompasses all of the above simplified routing models. This routing model is called a "Unified" Coefficient Routing Model. The term, unified, denotes the fact that it unifies many of the simplified routing models into a single model. The Unified Model also can serve as a framework in which the similarities and differences of the various models can be better understood. The Unified Model, although developed independently, is similar to a generalized kinematic model reported by Smith (1980).

The Unified Coefficient Routing Model has the following form:

$$O^{t+\Delta t} = C_1 I^t + C_2 I^{t+\Delta t} + C_3 O^t + C_4 \quad (1)$$

in which  $O$  is the outflow of a channel reach of length ( $\Delta x$ ),  $I$  is the inflow to the reach, the superscript ( $t$ ) denotes the variable evaluated at time ( $t$ ), the superscript ( $t+\Delta t$ ) denotes the variable evaluated at time ( $t+\Delta t$ ) where  $\Delta t$  is the time step; and  $C_1$ ,  $C_2$ , and  $C_3$  are routing coefficients which may be empirically derived or evaluated from the hydraulic characteristics of the channel reach. The coefficient  $C_4$  accounts for the effect of lateral inflows along the routing reach. Eq. (1) is called a "coefficient" routing model since the inflows ( $I^t$  and  $I^{t+\Delta t}$ ) and outflow ( $O^t$ ) are summed after multiplying each by a coefficient.

The variables,  $O$  and  $I$ , can also be represented by the discharge parameter,  $Q$ . Thus, Eq. (1) may take the following alternative form:

$$Q_{i+1}^{j+1} = C_1 Q_i^j + C_2 Q_i^{j+1} + C_3 Q_{i+1}^j + C_4 \quad (2)$$

in which the subscript ( $i$ ) denotes the upstream end of the routing reach, the subscript ( $i+1$ ) denotes the downstream end of the routing reach, the superscript ( $j$ ) denotes time ( $t$ ), and the superscript ( $j+1$ ) denotes time ( $t+\Delta t$ ).

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## STORAGE ROUTING MODELS

The Unified Coefficient Routing Model of the form of Eq. (1) will be developed first from the basic storage equation, i.e.,

$$\bar{I} - \bar{O} + \bar{q} \Delta x = \Delta s / \Delta t \quad (3)$$

$\bar{I}$  is the average inflow during the  $\Delta t$  time interval,  $\bar{O}$  is the average outflow during the time interval, and  $\bar{q}$  is the average lateral inflow (+) or outflow (-) along the  $\Delta x$  reach during the time interval. The storage (s) is assumed to be a function of inflow and/or outflow according to the following linear relation:

$$s = K [XI + (1-X) O] \quad (4)$$

in which X is a weighting coefficient,  $0 \leq X \leq 1$ .

Another weighting coefficient ( $\theta$ ) is introduced to regulate the relative importance of the parameters (I and O) at times (t) and (t+ $\Delta t$ ). Upon using Eq. (3), Eq. (4), and the  $\theta$  weighting factor, the following may be obtained:

$$\begin{aligned} \theta (I^{t+\Delta t} - O^{t+\Delta t}) + (1 - \theta) (I^t - O^t) + \bar{q} \Delta x = K/\Delta t [X (I^{t+\Delta t} \\ - I^t) + (1 - X) (O^{t+\Delta t} - O^t)] \end{aligned} \quad (5)$$

Then, using the relations,

$$K = \Delta x / c \quad \text{and} \quad \tilde{a} = c \Delta t / \Delta x = \frac{\Delta t}{K} \quad (6)$$

where c is the wave speed, Eq. (5) can be written as follows:

$$\begin{aligned} \theta I^{t+\Delta t} - \theta O^{t+\Delta t} + I^t - O^t - \theta I^t + \theta O^t + \bar{q} \Delta x = 1/\tilde{a} (X I^{t+\Delta t} \\ - X I^t + O^{t+\Delta t} - O^t - X O^{t+\Delta t} + X O^t) \end{aligned} \quad (7)$$

or

$$\begin{aligned} \theta \tilde{a} I^{t+\Delta t} - \theta \tilde{a} O^{t+\Delta t} + \tilde{a} I^t - \tilde{a} O^t - \theta \tilde{a} I^t + \theta \tilde{a} O^t + \tilde{a} \bar{q} \Delta x \\ - X I^{t+\Delta t} + X I^t - O^{t+\Delta t} + O^t + X O^{t+\Delta t} - X O^t = 0 \end{aligned} \quad (8)$$

Upon grouping like terms in Eq. (8), the following is obtained:

$$\begin{aligned} (-\theta \tilde{a} - 1 + X) O^{t+\Delta t} + (\tilde{a} - \theta \tilde{a} + X) I^t + (\theta \tilde{a} - X) I^{t+\Delta t} \\ + (-\tilde{a} + \theta \tilde{a} + 1 - X) O^t + \bar{q} \Delta x \tilde{a} = 0 \end{aligned} \quad (9)$$

After solving Eq. (9) for  $O^{t+\Delta t}$ , the following is obtained:

$$\begin{aligned} O^{t+\Delta t} = \left[ \frac{(1-\theta) \tilde{a} + X}{1 + \theta \tilde{a} - X} \right] I^t + \left[ \frac{\theta \tilde{a} - X}{1 + \theta \tilde{a} - X} \right] I^{t+\Delta t} \\ + \left[ \frac{1 - (1-\theta) \tilde{a} - X}{1 + \theta \tilde{a} - X} \right] O^t + \left[ \frac{\bar{q} \Delta x \tilde{a}}{1 + \theta \tilde{a} - X} \right] \end{aligned} \quad (10)$$

Eq. (10) can be placed in the same form as Eq. (1) if the coefficients ( $C_1$ ,  $C_2$ ,  $C_3$ , and  $C_4$ ) represent the leading coefficients in Eq. (10), i.e.,

$$C_1 = \frac{(1 - \theta) \tilde{a} + X}{1 + \theta \tilde{a} - X} \quad (11)$$

$$C_2 = \frac{\theta \tilde{a} - X}{1 + \theta \tilde{a} - X} \quad (12)$$

$$C_3 = \frac{1 - (1 - \theta) \tilde{a} - X}{1 + \theta \tilde{a} - X} \quad (13)$$

$$C_4 = \frac{\bar{q} \Delta x \tilde{a}}{1 + \theta \tilde{a} - X} \quad (14)$$

where:  $\tilde{a} = c \Delta t / \Delta x$  (15)

$$\bar{q} = (q^t + q^{t+\Delta t}) / 2 \quad (16)$$

$$0 \leq \theta \leq 1 \quad (17)$$

$$0 \leq X \leq 1 \quad (18)$$

#### KINEMATIC ROUTING MODEL

An equation of the form of Eq. (2) can be obtained also by using the classical kinematic wave equation. The kinematic equation is derived by using the conservation of mass equation, i.e.,

$$\partial Q / \partial x + \partial A / \partial t - q = 0 \quad (19)$$

and the assumption that the depth-discharge relation is single-valued, i.e.,

$$\partial A / \partial Q = dA / dQ = 1/c \quad (20)$$

in which  $c$  is the kinematic wave speed.

Then, since

$$\partial A / \partial t = \partial A / \partial Q \partial Q / \partial t = 1/c \partial Q / \partial t \quad (21)$$

Eq. (19) can be expressed as the classical kinematic wave equation, i.e.,

$$\partial Q / \partial x + 1/c \partial Q / \partial t - q = 0 \quad (22)$$

or 
$$\partial Q / \partial t + c \partial Q / \partial x - c q = 0 \quad (23)$$

Now, approximating Eq. (23) with finite differences which use  $X$  and  $\theta$  as weighting factors, the following is obtained:

$$\frac{X (Q_i^{j+1} - Q_i^j) + (1 - X) (Q_{i+1}^{j+1} - Q_{i+1}^j)}{\Delta t} + \frac{c}{\Delta x} [\theta (Q_{i+1}^{j+1} - Q_i^{j+1}) + (1 - \theta) (Q_{i+1}^j - Q_i^j)] - c \bar{q} = 0 \quad (24)$$

in which  $\bar{q} = (q_i^j + q_i^{j+1})/2$  and the subscript (i) denotes inflow, i+1 denotes outflow, the superscript (j) denotes time (t), and j+1 denotes time (t+Δt).

Eq. (24) can be rearranged into the following form:

$$X Q_i^{j+1} - X Q_i^j + Q_{i+1}^{j+1} - Q_{i+1}^j - X Q_{i+1}^{j+1} + X Q_{i+1}^j + c \Delta t / \Delta x (\theta Q_{i+1}^{j+1} - \theta Q_i^{j+1} + Q_{i+1}^j - Q_i^j - \theta Q_{i+1}^j + \theta Q_i^j) - c \bar{q} \Delta t = 0 \quad (25)$$

Letting  $\tilde{a} = c \Delta t / \Delta x$ , Eq. (25) can be rearranged as follows:

$$X Q_i^{j+1} - X Q_i^j + Q_{i+1}^{j+1} - Q_{i+1}^j - X Q_{i+1}^{j+1} + X Q_{i+1}^j + \theta \tilde{a} Q_{i+1}^{j+1} - \theta \tilde{a} Q_i^{j+1} + \tilde{a} Q_{i+1}^j - \tilde{a} Q_i^j - \theta \tilde{a} Q_{i+1}^j + \theta \tilde{a} Q_i^j - \bar{q} \tilde{a} \Delta x = 0 \quad (26)$$

Grouping similar terms in Eq. (26), gives the following:

$$(1 + \theta \tilde{a} - X) Q_{i+1}^{j+1} + (-X - \tilde{a} + \theta \tilde{a}) Q_i^{j+1} + (X - \theta \tilde{a}) Q_{i+1}^j + (-1 + X + \tilde{a} - \theta \tilde{a}) Q_{i+1}^j - \bar{q} \tilde{a} \Delta x = 0 \quad (27)$$

Upon solving Eq. (27) for  $Q_{i+1}^{j+1}$ , the outflow from the Δx reach at time (t+Δt), the following is obtained:

$$Q_{i+1}^{j+1} = \left[ \frac{(1 - \theta) \tilde{a} + X}{1 + \theta \tilde{a} - X} \right] Q_i^j + \left[ \frac{\theta \tilde{a} - X}{1 + \theta \tilde{a} - X} \right] Q_i^{j+1} + \left[ \frac{1 - (1 - \theta) \tilde{a} - X}{1 + \theta \tilde{a} - X} \right] Q_{i+1}^j + \left[ \frac{\bar{q} \Delta x \tilde{a}}{1 + \theta \tilde{a} - X} \right] \quad (28)$$

Finally, Eq. (28) can be placed in the same form as Eq. (2) if the coefficients ( $C_1, C_2, C_3$ , and  $C_4$ ) are defined by Eqs. (11-14).

#### UNIFIED COEFFICIENT ROUTING MODEL

Eqs. (10) and (28) are identical except for notation. The former is derived using the storage equation and the latter by using the kinematic wave equation. Herein, Eq. (28) is utilized as the Unified Coefficient Routing Model with coefficients  $C_1, C_2, C_3$ , and  $C_4$ . Thus,

$$Q_{i+1}^{j+1} = C_1 Q_i^j + C_2 Q_i^{j+1} + C_3 Q_{i+1}^j + C_4 \quad (29)$$

where:  $C_0 = 1 + \theta \tilde{a} - X \quad (30)$

$$C_1 = [(1 - \theta) \tilde{a} + X]/C_0 \quad (31)$$

$$C_2 = (\theta \tilde{a} - X)/C_0 \quad (32)$$

$$C_3 = [1 - (1 - \theta) \tilde{a} - X]/C_0 \quad (33)$$

$$C_4 = \bar{q} \Delta x \tilde{a}/C_0 \quad (34)$$

$$\tilde{a} = c \Delta t/\Delta x \quad (35)$$

$$\bar{q} = (q_i^j + q_i^{j+1})/2 \quad (36)$$

$$0 < \theta < 1 \quad (37)$$

$$0 < X < 1 \quad (38)$$

The various simple routing models can be obtained from the Unified Coefficient Routing Model, Eq. (29), by simply assigning particular values to the parameters ( $\theta$ ,  $X$ ,  $\tilde{a}$ ) of the coefficients,  $C_0$  through  $C_4$ , i.e.,

Muskingum:  $\theta = 1/2 \quad (39)$

$$0 < X < 1/2 \quad (40)$$

$$\tilde{a} = c\Delta t/\Delta x = \Delta x/K \Delta t/\Delta x = \Delta t/K \quad (41)$$

Reservoir:  $\theta = 1/2 \quad (42)$

$$X = 0 \quad (43)$$

$$\tilde{a} = c\Delta t/\Delta x = \Delta x/K \Delta t/\Delta x = \Delta t/K \quad (44)$$

Kinematic:  $1/2 < \theta < 1 \quad (45)$

$$0 < X < 1/2 \quad \left. \vphantom{0 < X < 1/2} \right\} \text{stable ranges, see fig. 1} \quad (46)$$

$$\tilde{a} = c\Delta t/\Delta x \quad (47)$$

Muskingum-Cunge:  $\theta = 1/2 \quad (48)$

$$X = 1/2 [1 - q_0/(c \Delta x S_0)] \quad (49)$$

$$\tilde{a} = c\Delta t/\Delta x \quad (50)$$

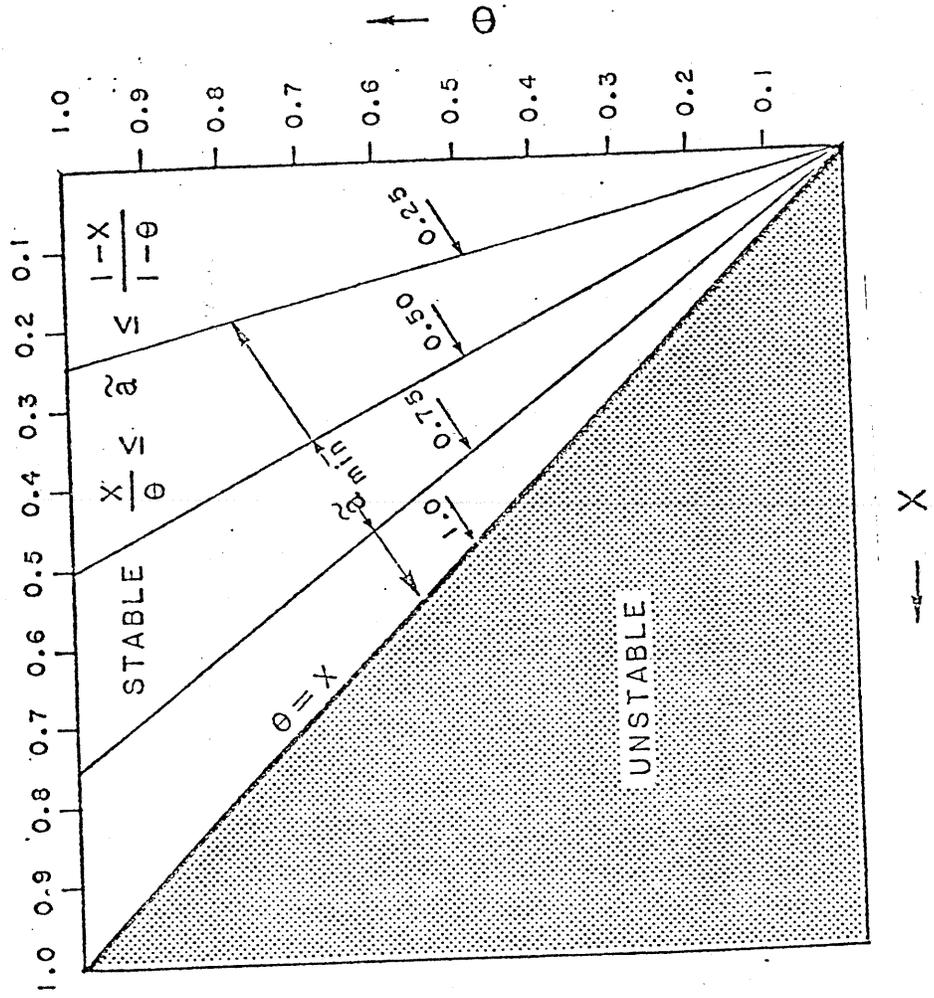


Figure 1. Stability of Kinematic Model as a function of  $X$  and  $\theta$  (after Smith, 1980).

SSAR:  $\theta = 1/2$  (51)

$X = 0$  (52)

$\tilde{a} = c\Delta t/\Delta x = \Delta x/K \Delta t/\Delta x = \Delta t/K = \Delta t/T_s$  (53)

Kalinin-Miljukov:  $\theta = 1/2$  (54)

$X = 0$  (55)

$\tilde{a} = c\Delta t/\Delta x = \Delta x/K \Delta t/\Delta x = \Delta t/K = \Delta t/\tau$  (56)

Lag and K:  $\theta = 1/2$  (57)

$X = 0$  (58)

$\tilde{a} = c\Delta t/\Delta x = \Delta x/K \Delta t/\Delta x = \Delta t/K$  (59)

SWMM:  $\theta = 0.55$  (60)

$X = 0.45$  (61)

$\tilde{a} = c\Delta t/\Delta x$  (62)

In the above equations, the parameters ( $q_o$ ,  $S_o$ ,  $T_s$ ,  $\tau$ ) are defined as unit-width discharge, channel bottom slope, time of storage, and time of propagation of a given discharge, respectively.

Verifications of the parameter values for  $\theta$ ,  $X$ , and  $\tilde{a}$  for the Muskingum, Reservoir, and SSAR models are given in Appendices A, B, and C, respectively. Since the Muskingum-Cunge Model is identical to the Muskingum Model except for the values of the parameters ( $X$  and  $\tilde{a}$ ), verification of the Muskingum Model is sufficient. The verification of the parameter values for the Kinematic Model is apparent from the derivation of Eq. (28). The verification of the Kalinin-Miljukov parameters is based on the fact that the Kalinin-Miljukov Model is identical to the Muskingum Model with  $X = 0$  (Miller and Cunge, 1975). The verification of the Lag and K Model is the same as the Reservoir Routing Model since the former is identical to the latter except the inflow is first lagged by either a constant value or some function of the inflow.

#### PARAMETER SELECTION

Proper evaluation of the parameters ( $c$ ,  $K$ ,  $\Delta t$ ,  $\Delta x$ ) is critical to successful application of the Unified Coefficient Routing Model.

Wave Speed (c). The kinematic wave speed ( $c$ ) may be computed from the following:

$$c = \beta V = 1.27 \beta S_o^{0.3} q_o^{0.4} / n^{0.6} \quad (63)$$

where:  $\beta = 1.67 - 0.67 A_o/B_o^2 \text{ dB}_o / dy$  (64)

in which  $q_0$  is the unit-width discharge,  $A_0$  is the associated cross-sectional area,  $B_0$  is the associated channel top width,  $dB_0/dy$  is rate of change of  $B_0$  with depth ( $y$ ),  $n$  is the Manning  $n$ , and  $S_0$  is the channel bottom slope. The parameters in Eqs. (63) and (64) may be assumed constant; in this case, they are associated with a reference discharge ( $Q_0$ ) which may be some characteristic flow such as the center of mass, the peak, or the mean of the discharge hydrograph. The parameters in Eqs. (63) and (64) may also be assumed time-dependent; in this case,  $c$  is continually re-evaluated at each time step. The first assumption produces a "linear" routing model, while the second produces a "nonlinear" model.

Storage Constant (K). The parameter (K) may be computed from the following:

$$K = \Delta x / c \quad (65)$$

in which  $\Delta x$  is the routing reach length and  $c$  is the wave speed.  $K$  may also be obtained empirically from discharge observations; it is equivalent to the time interval between the occurrence of the center of mass of inflow and that of the outflow.

Time Step ( $\Delta t$ ). The time step ( $\Delta t$ ) may be obtained from the following:

$$\Delta t \leq T_r / M \quad (66)$$

in which  $T_r$  is the time of rise of the inflow hydrograph and  $M$  is an integer with the following range of values:

$$6 \leq M \leq 20 \quad (67)$$

The larger values are associated with more rapid and nonuniform variation of the inflow hydrograph.

Reach Length ( $\Delta x$ ). The selection of the routing reach length ( $\Delta x$ ) affects the accuracy of the Muskingum-Cunge, Kalinin-Miljukov, and Kinematic Models. Ponce (1980) shows that  $\Delta x$  must be restricted to the following range for the Muskingum-Cunge Model:

$$\Delta x \leq 0.5 [c \Delta t + q_0 / (c S_0)] \quad (68)$$

The Kalinin-Miljukov Model requires the following reach length (Miller and Cunge, 1975):

$$\Delta x = \frac{Q_0}{S_0} \Delta y / \Delta Q \quad (69)$$

in which  $\Delta y / \Delta Q$  is the slope of the depth-discharge rating curve. The Kinematic Model requires the following reach length:

$$\Delta x \leq c \Delta t \quad (70)$$

Reach lengths for the other models (Muskingum, Reservoir, SSAR, Lag and K) are influenced by the values empirically obtained for the routing coefficients. Reach lengths that are much too long can have adverse effects on some of these models, e.g., the Muskingum Model.

Channel Slope ( $S_o$ ). The Muskingum-Cunge Model uses the channel slope ( $S_o$ ) in Eq. (49) to compute X and in Eq. (63) to compute the wave speed (c). If the Muskingum-Cunge Model is formulated as a linear model, X and c are constant with time although they may vary for each  $\Delta x$  reach. In this case,  $S_o$  is determined as the average energy slope within each  $\Delta x$  reach during the routing.  $S_o$  may be approximated as the average bottom slope; this is difficult to obtain in natural channels where the bottom slope is often quite irregular. The average slope can be obtained from the Manning equation, i.e.,

$$S_o = Q_o^2 B_o^{4/3} n / (2.21 A_o^{10/3}) \quad (71)$$

in which  $Q_o$  is the steady initial flow with associated top width ( $B_o$ ) and cross-sectional area ( $A_o$ ).

If the Muskingum-Cunge Model is formulated as a nonlinear model, X and c vary with time and  $\Delta x$ . In Eq. (49) and Eq. (63),  $q_o$ , B, and n vary with the flow. In this case,  $S_o$  may be obtained as in the linear model.

Depth-Discharge Relation. If the routing coefficients are evaluated from channel hydraulic characteristics, the variables such as c,  $q_o$ , A, B, etc. are dependent on a depth-discharge relation as given by the Manning equation, i.e.,

$$\tilde{Q} = 1.49 S_o^{1/2} \tilde{A}^{5/3} / (n \tilde{B}^{2/3}) \quad (72)$$

in which the symbol ( $\sim$ ) represents the average of the variable over time and along the  $\Delta x$  reach. Also, A and B must be known functions of the average depth ( $\tilde{y}$ ) in the reach as given by a table of  $\tilde{A}$  or  $\tilde{B}$  vs.  $\tilde{y}$  or by some appropriate analytical function such as:

$$\tilde{B} = k \tilde{y}^m \quad (73)$$

$$\tilde{A} = k \tilde{y}^{m+1} / (m+1) \quad (74)$$

in which k and m are fitted parameters representing cross-sectional scale and shape, respectively. Rectangular, parabolic, triangular shaped cross-sections have m values of 0, 0.5, and 1.0, respectively. A value of  $m > 1$  represents a  $\nabla$ -shaped section in which the width increases at a nonlinear rate with depth. In Eq. (70),  $\hat{Q}$  may be expressed in the following finite difference form:

$$\hat{Q} = 0.25 (Q_i^j + Q_i^{j+1} + Q_{i+1}^j + Q_{i+1}^{j+1}) \quad (75)$$

in which  $\hat{Q}$  is an estimated value.

Eq. (72) may be solved by an appropriate iterative technique such as Newton-Raphson. Hence, using Eqs. (72-75), the iterative expression for the average depth ( $\tilde{y}$ ) assumes the following form:

$$\tilde{y}^{\ell+1} = \tilde{y}^{\ell} - f(\tilde{y}^{\ell}) / [df(\tilde{y}^{\ell})/d\tilde{y}^{\ell}] \quad (76)$$

where:  $f(\tilde{y}^{\ell}) = \hat{Q} - 1.49 S_o^{1/2} \tilde{A}^{5/3} / (n \tilde{B}^{2/3})$  (77)

$$df(\tilde{y}^{\ell})/d\tilde{y}^{\ell} = -\hat{Q}(m + 5/3)/\tilde{y}^{\ell} \quad (78)$$

and  $l$  is the iteration counter. Convergence is attained when

$$|\tilde{y}^{\sim l+1} - \tilde{y}^{\sim l}| < \varepsilon_y \quad (79)$$

where  $\varepsilon_y$  is small depth such as 0.01 ft.

#### SOLUTION PROCEDURE

Linear Model. In the linear form of the Unified Coefficient Routing Model, the coefficients are considered to be constant for each  $\Delta x$  routing reach and throughout the duration of the routing computation. The appropriate coefficients in Eqs. (39-62) are evaluated initially from observations or the channel hydraulic characteristics. Then, Eq. (29) is applied recursively along each  $\Delta x$  routing reach and at each  $\Delta t$  time step until the routing is terminated.

Nonlinear Routing. In the nonlinear form of the Unified Coefficient Routing Model, the coefficients are considered to vary with each  $\Delta x$  routing reach and with time. The coefficients are evaluated explicitly, i.e., if they are dependent on the average depth ( $\tilde{y}$ ),  $\tilde{Q}$  in Eq. (72) is evaluated from Eq. (75) with the unknown discharge ( $\hat{Q}_{i+1}^{j+1}$ ) is estimated using a linearly extrapolated value. Thus,

$$\hat{Q}_{i+1}^{j+1} = Q_{i+1}^j + \Delta t^j / \Delta t^{j-1} (Q_{i+1}^j - Q_{i+1}^{j-1}) \quad (80)$$

Using Eq. (80) and Eq. (75),  $\tilde{Q}$  is evaluated and then  $\tilde{y}$  is computed from Eq. (76). Then,  $A$ ,  $B$  are computed from Eqs. (73-74) or a tabular function and the parameters  $q_0$ ,  $c$ ,  $X$ ,  $K$ , etc. are evaluated for the particular  $\Delta x$  reach and  $\Delta t^j$  time step. These parameters are used to compute the appropriate coefficients in Eqs. (39-62) which are used in Eq. (29) to compute the routed flow,  $Q_{i+1}^{j+1}$ . The procedure is then advanced to another routing reach or another time step, if there is only one routing reach, whenever

$$|Q_{i+1}^{j+1} - \hat{Q}_{i+1}^{j+1}| < \varepsilon_Q \quad (81)$$

where:  $\varepsilon_Q = \varepsilon_y \tilde{B} \tilde{Q} / \tilde{A}$  (82)

If the inequality, Eq. (81), is not satisfied, the next estimate ( $\hat{Q}_{i+1}^{j+1}$ ) is set equal to  $Q_{i+1}^{j+1}$  and the procedure is repeated. Provision may be made to specify  $\varepsilon_Q$  directly rather than using Eq. (82).

#### MODEL LIMITATIONS

The Unified Coefficient Routing Model is limited by accuracy constraints to the following applications: (1) insignificant backwater effects due to severe channel constrictions, bridges, dams, tributary inflows, tidal action; (2) negligible upstream wave propagation due to tides, storm surges, very large tributary inflow, and rapid operations of reservoir flow controls; and (3) specified ranges of the channel bottom slope ( $S_0$ ) and the time of rise ( $T_r$ ) of the inflow hydrograph.

The first two limitations arise from the model's inability to simulate upstream wave movement which encompasses backwater effects (small disturbances which propagate upstream) and the upstream propagation of waves.

The third limitation results from the inherent assumption of a single-valued depth-discharge relation, Eq. (20), used in the derivation of the Unified Coefficient Routing Model. This assumption produces the following constraint for all the variations of the Unified Coefficient Routing Model (Reservoir, Muskingum, Kinematic, SSARR, Lag and K, and SWMM) except the Muskingum-Cunge:

$$T_r S_o^{1.6} / (\Phi q_o^{0.2} n^{1.2}) \geq 0.2/E \quad (83)$$

where:  $\Phi = (m+1)^2 / (3m+5)$  (84)

and m is a cross-sectional shape parameter as used in Eq. (73); m = 0 for a rectangular or wide channel and  $\Phi = 0.2$ . E is the error (percent) that will be tolerated during the routing. The Muskingum-Cunge variation must satisfy the following:

$$T_r S_o^{0.7} n^{0.6} / (\Phi' q_o^{0.4}) \geq 0.003/E \quad (85)$$

where:  $\Phi' = (m+3) / (3m+5)$  (86)

Eq. (86) is for a prismatic channel.

Eq. (83) is for kinematic-type models and Eq. (85) is for diffusion-type models. A development of these criteria is given by Fread (1983).

#### CONCLUSIONS

A Unified Coefficient Routing Model is presented. It is derived from the storage routing equation and also from the classical kinematic wave equation. It is shown to unify various simplified routing models (Reservoir, Muskingum, Lag and K, Kinematic Wave, Kalinin-Miljukov, SSARR, Muskingum-Cunge, and SWMM) into a single model. Selection of parameters for the Unified Coefficient Routing Model is discussed and the model's limitations are identified.

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APPENDIX A

Verification of Unified Coefficients for Muskingum Model

The Muskingum Routing Model has the following basic form (Linsley et al., 1958; Miller and Cunge, 1975; Ponce and Yevjevich, 1978):

$$Q_{i+1}^{j+1} = C'_1 Q_i^j + C'_2 Q_i^{j+1} + C'_3 Q_{i+1}^j \quad (\text{A-1})$$

where:  $C'_0 = K - KX + \Delta t/2 \quad (\text{A-2})$

$$C'_1 = (KX + \Delta t/2)/C'_0 \quad (\text{A-3})$$

$$C'_2 = (-KX + \Delta t/2)/C'_0 \quad (\text{A-4})$$

$$C'_3 = (K - KX - \Delta t/2)/C'_0 \quad (\text{A-5})$$

Using Eqs. (39-41) to evaluate  $\theta$ ,  $X$ , and  $\tilde{a}$  in Eqs. (30-35) for the coefficients ( $C_0, C_1, C_2, C_3$ ), the following are obtained:

$$C_0 = 1 + \theta \tilde{a} - X = 1 + 0.5 \Delta t/K - X = (K - KX + \Delta t/2)/K = C'_0/K \quad (\text{A-6})$$

$$C_1 = \frac{(1-\theta)\tilde{a} + X}{1 + \theta \tilde{a} - X} = \frac{(1-0.5)\Delta t/K + X}{C'_0/K} = \frac{(KX+\Delta t/2)/K}{C'_0/K} = C'_1 \quad (\text{A-7})$$

$$C_2 = \frac{\theta \tilde{a} - X}{1 + \theta \tilde{a} - X} = \frac{0.5 \Delta t/K - X}{C'_0/K} = \frac{(-KX+\Delta t/2)/K}{C'_0/K} = C'_2 \quad (\text{A-8})$$

$$C_3 = \frac{1 - (1-\theta)\tilde{a} - X}{1 + \theta \tilde{a} - X} = \frac{1 - (1-0.5)\Delta t/K - X}{C'_0/K} = \frac{(K-KX-\Delta t/2)/K}{C'_0/K} = C'_3 \quad (\text{A-9})$$

Since  $C_1, C_2, C_3$  are identical to  $C'_1, C'_2, C'_3$ , respectively, the verification of Eqs. (39-41) is achieved; and the Unified Coefficient Routing Model, Eq. (29), is identical to the Muskingum Model, Eq. (A-1) through Eq. (A-5).

APPENDIX B

Verification of Unified Coefficients for Reservoir Model

The Reservoir Routing Model may be developed from the following expression (Linsley, et al., 1958):

$$\bar{I} - \bar{O} = \Delta(KO)/\Delta t \quad (B-1)$$

Upon expanding Eq. (B-1), the following is obtained:

$$0.5 (I^t + I^{t+\Delta t}) - 0.5 (O^t + O^{t+\Delta t}) = (KO^{t+\Delta t} - KO^t)/\Delta t \quad (B-2)$$

Rearranging Eq. (B-2) gives:

$$O^{t+\Delta t} = \left(\frac{\Delta t/2}{K+\Delta t/2}\right) I^t + \left(\frac{\Delta t/2}{K+\Delta t/2}\right) I^{t+\Delta t} + \left(\frac{K-\Delta t/2}{K+\Delta t/2}\right) O^t \quad (B-3)$$

Therefore, the Reservoir Routing Model may be expressed as follows:

$$O^{t+\Delta t} = C'_1 I^t + C'_2 I^{t+\Delta t} + C'_3 O^t \quad (B-4)$$

where:  $C'_0 = K + \Delta t/2$  (B-5)

$$C'_1 = (\Delta t/2)/C'_0 \quad (B-6)$$

$$C'_2 = (\Delta t/2)/C'_0 \quad (B-7)$$

$$C'_3 = (K - \Delta t/2)/C'_0 \quad (B-8)$$

Using Eqs. (42-44) to evaluate  $\theta$ ,  $X$ , and  $\tilde{a}$  in Eqs. (30-35) for the coefficients ( $C_0, C_1, C_2, C_3$ ), the following are obtained:

$$C_0 = 1 + \theta \tilde{a} - X = 1 + 0.5 \Delta t/K = (K + \Delta t/2)/K = C'_0/K \quad (B-9)$$

$$C_1 = \frac{(1-\theta)\tilde{a} + X}{1 + \theta \tilde{a} - X} = \frac{(1-0.5)\Delta t/K}{C'_0/K} = \frac{(\Delta t/2)/K}{C'_0/K} = C'_1 \quad (B-10)$$

$$C_2 = \frac{\theta \tilde{a} - X}{1 + \theta \tilde{a} - X} = \frac{0.5 \Delta t/K}{C'_0/K} = \frac{(\Delta t/2)/K}{C'_0/K} = C'_2 \quad (B-11)$$

$$C_3 = \frac{1 - (1-\theta)\tilde{a} - X}{1 + \theta \tilde{a} - X} = \frac{1 - (1-0.5)\Delta t/K}{C'_0/K} = \frac{(K-\Delta t/2)/K}{C'_0/K} = C'_3 \quad (B-12)$$

Since  $C_1, C_2, C_3$  are identical to  $C'_1, C'_2, C'_3$ , respectively, the verification of Eqs. (42-44) is achieved. Thus, the Unified Coefficient Model is identical to the Reservoir Routing Model, Eq. (B-4) through Eq. B-8).

APPENDIX C

Verification of Unified Coefficients for SSAR Model

The SSAR Model (Rockwood, 1958; Miller and Cunge, 1975) may be expressed in the following form:

$$O^{t+\Delta t} = \left( \frac{\bar{I} - O^t}{T_s + \Delta t/2} \right) \Delta t + O^t \quad (C-1)$$

in which  $T_s$  is a storage constant. After expanding Eq. (C-1) and grouping like terms, the following coefficient form of Eq. (C-1) is obtained:

$$O^{t+\Delta t} = C'_1 I^t + C'_2 I^{t+\Delta t} + C'_3 O^t \quad (C-2)$$

where:  $C'_0 = T_s + \Delta t/2$  (C-3)

$$C'_1 = (\Delta t/2)/C'_0 \quad (C-4)$$

$$C'_2 = (\Delta t/2)/C'_0 \quad (C-5)$$

$$C'_3 = (T_s - \Delta t/2)/C'_0 \quad (C-6)$$

Using Eqs. (51-53) to evaluate  $\theta$ ,  $X$ , and  $\tilde{a}$  in Eq. (30-34) for the coefficients ( $C_0, C_1, C_2, C_3$ ), the following are obtained:

$$C_0 = 1 + \theta \tilde{a} - X = 1 + 0.5 \Delta t/T_s = (T_s + \Delta t/2)/T_s = C'_0/T_s \quad (C-7)$$

$$C_1 = \frac{(1-\theta)\tilde{a} + X}{1 + \theta \tilde{a} - X} = \frac{(1-0.5)\Delta t/T_s}{C'_0/T_s} = \frac{(\Delta t/2)T_s}{C'_0/T_s} = C'_1 \quad (C-8)$$

$$C_2 = \frac{\theta \tilde{a} - X}{1 + \theta \tilde{a} - X} = \frac{0.5 \Delta t/T_s}{C'_0/T_s} = \frac{(\Delta t/2)/T_s}{C'_0/T_s} = C'_2 \quad (C-9)$$

$$C_3 = \frac{1 - (1-\theta)\tilde{a} - X}{1 + \theta \tilde{a} - X} = \frac{1 - (1-0.5)\Delta t/T_s}{C'_0/T_s} = \frac{(T_s - \Delta t/2)/T_s}{C'_0/T_s} = C'_3 \quad (C-10)$$

Since  $C_1, C_2, C_3$  are identical to  $C'_1, C'_2, C'_3$ , respectively, the verification of Eqs. (51-53) is achieved. Thus, the Unified Coefficient Routing Model is identical to the SSAR Routing Model, Eq. (C-2) through Eq. (C-6).

Prismatic Channel Routing Comparison Program  
Data Input

<u>Card Input No.</u>	<u>Description</u>
(1)	CL,SOMI,CCK,CCM,CCN,DI,DBG - 7F10.2
	CL            Length of channel, mi
	SOMI         Slope of channel bottom, ft/mi
	CCK          Cross-section parameter, scale factor, k
	CCM          Cross-section parameter, shape factor, m
	CCN          Manning n (constant with depth)
	DI            Initial flow depth, ft
	DBG          parameter for controlling output for debug; DBG=0.0, minimum output; DBG=1.0, medium level of output; DBG=2.0, maximum level of output
<u>NOTE:</u>	CCK,CCM,CCN, and DI are left blank if channel is <u>not</u> described by B=kD**m relation, in which case cards (2), (3), and (4) must be read in; otherwise, cards (2), (3), and (4) are omitted.
(2)	CMI1,CMI2,CMI3,CMI4 - 4F10.3
	CMI1         Manning n for channel depth = 0.0
	CMI2         Manning n for channel depth = DI1
	CMI3         Manning n for channel depth = DI2+DI1
	CMI4         Manning n for channel depth = DI3+DI2+DI1
(3)	BI1,BI2,BI3,BI4 - 4F10.2
	BI1          Channel width(ft) for depth = 0.0
	BI2          Channel width(ft) for depth = DI1
	BI3          Channel width(ft) for depth = DI2+DI1
	BI4          Channel width(ft) for depth = DI3+DI2+DI1
(4)	DI1,DI2,DI3 - 3F10.2
	DI1          Incremental depth(ft) from channel bottom to BI2
	DI2          Incremental depth(ft) from BI2 to BI3
	DI3          Incremental depth(ft) from BI3 to BI4

(5) GAMA,RHO,TEH - 3F10.2

GAMA Hydrograph parameter (skewness, TCG/TPH),  
dimensionless  
RHO Ratio of peak flow to initial flow (QP/QO),  
dimensionless  
TEH End of routing period, hrs (may be left blank, in  
which case TEH is computed by program as  
TEH=(3+TPH/6.)\*TPH+CL/(2 SQRT(SOMI)).

(6) NROUT,NDB,MMKI - 3I10

NROUT Total number of routing models being run  
NDB Parameter indicating whether the last cross-section  
will be used to compare routing models; NDB=0, no;  
NDB=1, yes.  
MMKI Number of values in table of K vs Q for Muskingum  
Routing

(7) KKR(L),TPH,FQV,DXI,DTH,X,TH,FKKK - I10,7F10.2

KKR(L) Code number for type of routing (See following Note)  
TPH Time (hrs) to peak of inflow hydrograph (gamma  
function)  
FQV Ratio of selected flow to peak flow (selected flow is  
for linearization of routing coefficients); if FQV is  
left blank, it is assumed to be 1.0  
DXI Delta x reach length (mi); must be evenly divisible  
into CL (the channel length)  
DTH Delta t time step (hr); DTH=TPH/MM where MM varies  
from 6 to 20  
X Weighting parameter in routing models (dimensionless),  
may be left blank for certain routing models  
TH Weighting parameter in routing models (dimensionless),  
usually set equal to 0.5  
FKKK K storage factor (hrs); constant for all flows; may be  
left blank unless required and independent of flow

NOTE: L index goes from 1 to NROUT; card (7) is read in for each L

NOTE: KKR(L) = 1 -- Dynamic  
= 2 -- Muskingum - Cunge (Linear)  
= 3 -- Muskingum-Cunge (Nonlinear, iterative)  
= 4 -- Muskingum-Cunge (Nonlinear, iterative)  
= 5 -- Dynamic-Muskingum-Cunge  
= 6 -- Kinematic  
= 7 -- Kinematic (Iterative)  
= 8 -- Muskingum, Reservoir (constant k)  
= 9 -- Muskingum (variable k)  
=10 -- Diffusion

(8) FMK(L) - 8F10.2

FMK(L) Storage constant, K, (hrs); L index goes from 1 to  
MMKI

(9) FMKIN(L) - 8F10.2

FMKIN(L) Flow ( $\text{ft}^3/\text{sec}$ ) associated with FMK(L); L index goes  
from 1 to MMKI

NOTE: Cards (8) and (9) are read in only if MMKI.GE.2 and KKRN(L)=9

(10) IRT(L) - 8I10

IRT(L) Code number for each cross-section; if IRT(L)=0,  
Dynamic routing is used; if IRT(L)=1, Muskingum-Cunge  
Routing is used; L index goes from 1 to N-1, where  
N=CL/DXI

NOTE: Card (10) is read in only if KKRN(L)=5