# *Development and Testing of Snow Pack Energy Balance Equations*

E. A. ANDERSON

*Environmental Science Services Administration*
*Weather Bureau, Office of Hydrology, Washington, D. C.*

*Abstract.* By making several minor assumptions and using an empirical technique in one situation, the energy balance of a snow pack can be computed on a continuous basis. Net radiation or its components, air temperature, dew-point temperature, atmospheric pressure, amount of precipitation and its temperature, surface snow density, and a wind function that relates vapor pressure gradient to moisture transfer as a function of wind speed, must be either measured or estimated. Comparison of computed versus observed snow pack runoff and snow surface temperature for the Lower Meadow lysimeter studies of 1954 at the Central Sierra Snow Laboratory reveals good agreement. A model was constructed by combining the energy balance equations with relationships describing the other components of the snow accumulation and ablation processes. The Stanford Watershed Model is used to determine the time delay from when water leaves the snow pack to the streamflow gaging station. Tests of the model on 5 years of data from the Central Sierra Snow Laboratory resulted in a reasonable simulation of the observed mean daily flow hydrograph and seasonal areal snow cover plot. It is concluded that the model is at least a good approximation to the actual physical processes and should be a valuable tool in further snow hydrology investigations. (Key words: Snow; synthesis; energy balance)

## SYMBOLS

$A$, albedo;
$A_i$, albedo index;
$C_e$, corrects point vapor pressure measurement to average watershed elevation;
$E$, amount of moisture transferred to snow pack by turbulent exchange process;
$K$, $= K' \cdot K''$;
$K'$ empirical constant adjusting wind from point to areal value;
$K''$, empirical wind function constant at a point;
$K_H$, sensible heat eddy transfer coefficient;
$K_s$, effective thermal conductivity in cal $gm^{-1}$ $sec^{-1}$ $°C^{-1}$;
$K_W$, water vapor eddy transfer coefficient;
$L$, latent heat of vaporization in cal/gm;
$L_v$, latent heat of vaporization in cal/$cm^2$/inch of water;
$L_s$, latent heat of sublimation in cal/$cm^2$/inch of water;
$M$, snowmelt;
$P$, amount of new precipitation;
$P_a$, atmospheric pressure;
$Q_a$, incident terrestrial radiation;
$Q_c$, heat transferred to or from the surface by conduction within the snow pack;
$Q_e$, gain or loss of latent heat caused by evaporation, condensation, or sublimation;
$Q_h$, sensible heat transfer;
$Q_i$, incident solar radiation;
$Q_l$, gain or loss of latent heat caused by melting or freezing;
$Q'_l$, gain or loss of latent heat caused by freezing or melting in the surface layer of snow pack;
$Q_m'$, heat causing surface layer melting;
$Q_r$, net radiation heat transfer;
$Q_s$, net transfer of heat by conduction at the snow-ground interface;
$Q_w$, heat content of a gain or loss in the water equivalent of the snow pack;
$Q_w'$, heat content of a gain or loss in the water equivalent of the surface layer;
$Q_{wr}$, heat content of rain;
$Q_{ws}$, heat content of new snow;
$Q_\theta$, heat storage of the snow pack;
$R$, routing coefficient for transient water in the snow pack;
$S$, amount of new snow;
$S_b$, water equivalent before new snow;
$S_i$, index value of watershed snow pack water equivalent;
$T_a$, temperature of air;
$T_d$, dewpoint temperature;
$T_p$, temperature of new precipitation;
$T_s$, snow pack surface temperature;
$T_{sk}$, snow surface temperature in °K;
$T_w$, wet-bulb temperature;
$W$, portion of $S$ that must melt before partial-cover condition again exists;
$a$, coefficient used to calculate vapor pressure from temperature;

$b$, coefficient used to calculate vapor pressure from temperature;
$c$, coefficient used to calculate back radiation from snow;
$c_i$, specific heat of ice;
$c_p$, specific heat of dry air;
$c_w$, specific heat of water;
$d$, coefficient used to calculate back radiation from snow;
$e$, vapor pressure;
$e_a$, vapor pressure of air;
$e_s$, vapor pressure at snow surface;
$f(u)$, wind function for water vapor transfer;
$f(u)'$, wind function for sensible heat transfer;
$f(u_w)$, wind function relating wind at a point to wind over a watershed;
$g$, empirical constant used in determining albedo;
$m$, empirical constant used in determining albedo;
$n$, empirical constant used in determining albedo;
$q$, specific humidity;
$t$, time in seconds;
$u$, wind velocity in miles traveled;
$z$, height above the surface;
$\beta$, Bowen's ratio;
$\gamma$, dimensional coefficient;
$\rho$, air density;
$\rho_s$, density of snow in water equivalent per unit volume;
$\rho_s'$, density of surface layer of snow pack;
$\sigma$, Stefan-Boltzmann constant ($11.71 \times 10^{-8}$ langleys/day$^{-1}$ °K$^{-4}$).

## INTRODUCTION

Previous work by *Anderson and Crawford* [1964] at Stanford University showed that it was feasible to model the snow accumulation and melt processes. Two approaches were tested: an empirical approach using air temperature as the sole index to the heat exchange between the snow and the atmosphere, and a simple energy balance method to determine the heat gained by the snow pack during periods of melt. The present Stanford Watershed Model as developed by *Crawford and Linsley* [1966] uses a combination of these two approaches.

Subsequent work by the author at the Weather Bureau has been concerned with the development of a more physically rational model of the snow accumulation and melt processes by utilizing the current knowledge of snow hydrology. This paper describes the development and testing of such a model.

## DEVELOPMENT OF THE SNOW PACK ENERGY BALANCE EQUATIONS

The energy balance of a snow pack for any time increment can be expressed as

$$Q_\theta = Q_r + Q_h + Q_e + Q_l + Q_s + Q_w \quad (1)$$

where $Q_\theta$ is the change in the heat storage of the snow pack, $Q_r$ is the net radiation heat transfer, $Q_h$ is the sensible heat transfer, $Q_e$ is the gain or loss of latent heat caused by evaporation, condensation, or sublimation, $Q_l$ is the gain or loss of latent heat caused by freezing or melting, $Q_s$ is the net transfer of heat by conduction at the snow-ground interface, and $Q_w$ is the net heat transfer by a gain or loss of water. A positive sign indicates a gain in heat by the snow.

If $Q_s$ is temporarily ignored, the remaining energy transformations take place in the surface layer of the snow pack with three exceptions: the heat content of snow pack runoff, the gain of latent heat caused by freezing of liquid water within the snow pack, and the gain of heat caused by the penetration of solar radiation. Since water leaving the snow is generally 32°F, the first exception can be ignored if 32°F is used as the zero point for computations of heat storage.

If it is assumed that none of the solar radiation penetrates the surface layer of the snow pack, and that its heat storage is negligible, the energy balance of this layer can be expressed as

$$Q_r + Q_h + Q_e + Q_l' + Q_c + Q_w' = 0 \quad (2)$$

where $Q_l'$ is the gain or loss of latent heat caused by freezing or melting in the surface layer, $Q_c$ is the heat transferred to or from this layer by conduction within the snow pack, and $Q_w'$ is the heat transferred by a gain or loss in water by the surface layer. Since water leaving this layer is generally 32°F, $Q_w'$ is the heat transferred to the snow by precipitation. Thus $Q_w' = Q_w$. In addition, since the amount of liquid water available for freezing in the surface layer is generally conceded to be small, it is neglected. Thus $Q_l'$ represents only the loss of latent heat caused by melting.

Although no specific thickness will be designated for this surface layer, it can be thought of as the upper 1 or 2 inches of the snow pack. The assumption of complete absorption of solar radiation in this layer will of course have no

effect at night. During daytime periods, studies by *Gerdel* [1948] have shown that 80% of the solar radiation absorbed by the snow is absorbed in the top 2 and 6 inches for densities of 26 and 43%, respectively. If the snow pack is isothermal at 32°F, as during active melt, it makes no difference, in the calculation of melt, where solar radiation is absorbed, as long as the snow is deep enough so that significant solar energy does not penetrate to the soil. Thus the assumption of complete solar absorption in the surface layer can only result in an error during daytime periods when the snow pack is not isothermal at 32°F. This condition is more likely to occur during the accumulation period (although it can also occur during cold spells during the melt season) when surface density and solar radiation intensity are less than during the active melting period. Based on these considerations, it is believed that the error caused by the assumption of complete solar absorption in the surface layer is generally small.

Before proceeding further with the snow pack energy balance equations, it is necessary to examine some of the components separately.

*Latent heat transfer.* The phase changes involved in the transfer of water vapor to or from the snow pack vary with the condition at the snow surface. If the snow is melting, evaporation or condensation, depending on the direction of the vapor pressure gradient, takes place. However, if the snow is not melting, there is a phase change from vapor to solid or vice versa. The vapor-liquid phase change differs from the vapor-solid change by the heat of fusion. Therefore, during periods of melt the latent heat of vaporization or condensation is applicable in energy transfer computations, whereas during remaining periods the latent heat of sublimation should be used.

The transfer of moisture to or from the snow is a turbulent exchange process whose direction is controlled by the vapor pressure gradient. When the vapor pressure in the air is greater than the vapor pressure at the surface, there is a transfer of moisture from the air to the snow that results in a release of latent heat. When the gradient is reversed, the transfer of moisture and heat is also reversed. This transfer of moisture can be simply expressed as

$$E = f(u) \cdot (e_a - e_s) \tag{3}$$

where $E$ is the amount of moisture transferred, $f(u)$ is the appropriate wind function, $e_a$ is the vapor pressure of the air, and $e_s$ is the vapor pressure at the snow surface. It is usually assumed that $e_s$ is equal to the saturation vapor pressure at the temperature of the snow surface. $Q_e$ can be determined from $E$ by applying the appropriate latent heat as discussed in the preceding paragraph.

*Sensible heat transfer.* The transfer of sensible heat from the air to the snow or vice versa is also a turbulent exchange process. Its direction is controlled by the direction of the temperature gradient. When the overlying air is warmer than the snow surface, there is a direct transfer of heat from the air to the snow and vice versa. The transfer of sensible heat can be simply expressed, similarly to the transfer of moisture, as

$$Q_h = f(u)' \cdot (T_a - T_s) \tag{4}$$

where $f(u)'$ is the appropriate wind function, $T_a$ is the temperature of the air, and $T_s$ is the surface temperature of the snow pack.

If measurements are made at the surface of the gain or loss of moisture, air temperature, snow temperature, and wind velocity, $f(u)$ can be determined for that measurement site. Thus $Q_e$ can be computed. However, $Q_h$ cannot be measured directly, except as the residual of the energy balance. Measurements with sufficient reliability to calculate $Q_h$ with reasonable accuracy have not been made. An alternate method for determining $Q_h$ is the use of the *Bowen* ratio [1926], the ratio of $Q_h$ to $Q_e$. Bowen's ratio ($\beta$) can be expressed as

$$\beta = Q_h/Q_e = -\rho \cdot C_p \cdot K_H \cdot (\partial T_a/\partial z)/-L \cdot \rho \cdot K_W \cdot (\partial q/\partial z) \tag{5}$$

where $\rho$ is the air density, $c_p$ the specific heat of dry air, $L$ the latent heat of vaporization, $K_w$ the eddy transfer coefficient for water vapor, $K_H$ the eddy transfer coefficient for heat, $T_a$ air temperature, $q$ specific humidity, and $z$ the height above the surface. An approximation of $\beta$ can be obtained from

$$\beta = \gamma \cdot (T_a - T_s)/(e_a - e_s) \tag{6}$$

assuming that $K_H$ and $K_W$ are equal. The term $\gamma$ can be determined by using the relationship

between specific humidity, vapor pressure ($e$), and atmospheric pressure ($P_a$), which is

$$q = 0.622 \cdot e/P_a \tag{7}$$

Thus $\gamma$ can be expressed as

$$\gamma = C_p \cdot P_a/0.622 \cdot L \tag{8}$$

If $C_p$ is taken as 0.133 gal/gm$^{-1}$ °F$^{-1}$ and $L$ as 597.3 cal/gm, its value at 32°F, $\gamma$ becomes 0.00036 $P_a$. When $P_a$ has the unit of inches of mercury, $\gamma$ has the unit of inches of mercury per °F.

As summarized by *Pruitt and Lourence* [1966], the use of the Bowen ratio to compute evaporation for time periods as short as 15 minutes has given highly favorable results, indicating the validity of the assumption of the equality of $K_H$ and $K_W$. The only atmospheric condition under which doubt still seems to exist as to the validity of this assumption is the situation of high instability. Because this situation does not frequently occur over a snow pack, and considering time periods of 12 hours, this assumption should be valid for the computations described later in this paper.

*Precipitation heat transfer.* The heat content of new precipitation ($Q_{ws}$ for snow and $Q_{wr}$ for rain) can be determined, if the temperature ($T_p$), type, and amount ($P$) of the precipitation are known, by the expressions

$$\text{snow} \qquad Q_{ws} = P \cdot T_p c_i \tag{9}$$

$$\text{rain} \qquad Q_{wr} = P \cdot T_p c_w \tag{10}$$

where $c_i$ is the specific heat of ice, and $c_w$ is the specific heat of water. Since the temperature of the precipitation is usually not measured, it is commonly assumed to be equal to the wet-bulb temperature ($T_w$).

*Conductive heat transfer.* The most difficult portion of equation 2 to evaluate is the conductive heat transfer. *Yen* [1962] summarized the results of various investigators on the variation in the thermal conductivity of snow with density. The relationships show some variation. Herein the relationship determined by *Yen* [1965], which is close to the average of the other determinations, is used. This relationship can be expressed as

$$K_s = 0.0077 \cdot \rho_s^{\,2} \tag{11}$$

where $K_s$ is the effective thermal conductivity in cal/gm$^{-1}$ sec$^{-1}$ °C$^{-1}$, and $\rho_s$ is the density of the snow in water equivalent per unit volume. It should be pointed out that the thermal conductivity of snow, even at a density of 0.5, is small. Therefore, the amount of heat transferred by conduction is minor, compared with the heat transferred between the surface layer and the atmosphere. Thus detailed measurements or computations of this term on a continuous basis are not generally warranted.

Even though the determination of conductive heat transfer on a continuous basis is not warranted, there is one condition under which a reasonable approximation of this term seems justified and can easily be made. This is the case of the cooling of an isothermal snow pack following a period of melt. In this situation Fourier's expression for the loss of heat from an isothermal medium with a constant temperature applied at the surface, as suggested by *Wilson* [1941], can be used. Even though the snow surface temperature varies, using the average $T_s$ should result in a good estimate over a period such as a night. The conductive transfer under this situation is

$$Q_c = 2 \cdot K_s \cdot (T_s - 32.0) \cdot \sqrt{t} / \sqrt{K_s \cdot \pi / C_i \cdot \rho_s'} \tag{12}$$

where $t$ is time in seconds, and $\rho_s'$ is the density of the surface layer of the snow pack.

The use of the $K_s$ of equation 11 carries with it the assumption that the thermal conductivity of snow varies only with density. Whether the snow is wet or dry undoubtedly also has an effect. *Gerdel* [1954] reported that in a majority of measurements during the melt season the residual liquid-water content was less than 2% of the water equivalent of the snow. Based on this amount, it seems likely that changes in the thermal conductivity caused by the presence of liquid water are small. However, when portions of the snow pack where liquid water exists are cooled sufficiently, this liquid water will freeze, releasing its latent heat of fusion. Thus the snow surface temperature calculated from equation 12 should be slightly lower than that observed.

To solve equation 2 for $Q_i'$ and/or $Q_c$, certain of the variables must be measured or estimated. These are $Q_r$, $f(u)$, $e_a$, $T_a$, $P_a$, $P$, $T_w$, and $\rho_s'$. If net radiation data are not available, but observations or estimates of the other components of

the radiation balance are, $Q_r$ can be expressed as

$$Q_r = Q_i \cdot (1.0 - A) + Q_a - \sigma \cdot T_{sk}^4 \qquad (13)$$

where $Q_i$ is the incident solar radiation, $A$ is the albedo, $Q_a$ is the incident terrestrial radiation, $\sigma$ is the Stefan-Boltzmann constant ($11.71 \cdot 10^{-8}$ langleys/day$^{-1}$ °K$^{-4}$), and $T_{sk}$ is the snow surface temperature in °K.

In two situations equation 2 can be solved. The first is the case of melt from a snow pack isothermal at 32°F; thus $T_s = 32$°F and $Q_c = 0$. Replacing $-Q_1'$ by $Q_m'$, the heat causing melt, and since any precipitation occurring during this situation would be likely to be rain, equation 2 can now be expressed as

$$Q_m' = Q_r + Q_h + Q_e + Q_{wr} \qquad (14)$$

Substituting from equations 3, 4, 6, and 10, the heat available for melt becomes

$$Q_m' = Q_r + L_v \cdot f(u) \cdot [\gamma \cdot (T_a - T_s) + (e_a - e_s)] + P \cdot c_w \cdot (T_p - T_s) \qquad (15)$$

where $L_v$ is the latent heat of vaporization. Using units of calories cm$^{-2}$ for heat, inches for water equivalent, °F for temperature, and inches of mercury for pressure, equation 15 becomes

$$Q_m' = Q_r + 1517.0 \cdot f(u) \cdot [0.00036 \cdot P_a \cdot (T_a - 32.0) + (e_a - 0.1804)] + 1.41 \cdot P \cdot (T_w - 32.0) \qquad (16)$$

Since it takes 202.4 calories cm$^{-2}$ to melt one inch water equivalent of snow at 32°F, the snowmelt ($M$) in inches is

$$M = Q_m'/202.4 \qquad (17)$$

The second situation when equation 2 can be solved is during a period with no melt immediately following a time when the snow pack was isothermal at 32°F (basically nighttime cooling during the melt season). In this case $Q_1' = 0$, and $Q_c$ can be expressed by equation 12. Substituting from equations 3, 4, 6, and 9 or 10, equation 2 becomes

$$Q_r + L_s \cdot f(u) \cdot [\gamma \cdot (T_a - T_s) + (e_a - e_s)] - 2 \cdot K_s \cdot (T_s - 32.0) \cdot \sqrt{t} / \sqrt{K_s \cdot \pi / C_i \cdot \rho_s'} + Q_w' = 0 \qquad (18)$$

where $L_s$ is the latent heat of sublimation. Since the relationship to calculate saturation vapor pressure from temperature involves temperature to a power much greater than 1 or as an exponent of the Naperian base, an exact solution of equation 18 for $T_s$ would be time consuming. However, if $e_s$ is set equal to $a + b \cdot T_s$, various temperature intervals can be selected so that $T_s$ can be computed directly within 0.5°F. Making this substitution, the equation for $T_s$ becomes

$$T_s = \{Q_r + L_s \cdot f(u) \cdot (\gamma \cdot T_a + e_a - a) + [64 \cdot K_s \cdot \sqrt{t} / \sqrt{K_s \cdot \pi / c_i \cdot \rho_s'}] + Q_w'\} / [L_s \cdot f(u) \cdot (\gamma + b) + 2 \cdot K_s \cdot \sqrt{t} / \sqrt{K_s \cdot \pi / c_i \cdot \rho_s'}] \qquad (19)$$

Using the same units as for equation 16, substituting further from equation 11, and considering a duration of 12 hours, the equation for snow surface temperature in this situation is of the form

$$T_s = [Q_r + (0.00036 \cdot P_a \cdot T_a + e_a - a) \cdot 1719.5 \cdot f(u) + 260 \cdot \rho_s'^{3/2} + Q_w'] / [(0.00036 \cdot P_a + b) \cdot 1719.5 \cdot f(u) + 8.1 \cdot \rho_s'^{3/2}] \qquad (20)$$

where $a$ and $b$ have the values

| | | |
|---|---|---|
| $20°\text{F} \leq T_s \leq 32°\text{F}$ | $a = 0.0175$ | $b = 0.00618$ |
| $0°\text{F} \leq T_s < 20°\text{F}$ | $a = -0.042$ | $b = 0.0032$ |
| $T_s < 0°\text{F}$ | $a = -0.043$ | $b = 0.0014$ |

If $Q_r$ is expressed as in equation 13, the form of the equation changes slightly. An approximation of $\sigma \cdot T_{sk}^4$ can be made in the form $c + d \cdot T_s$, so that equation 20 can be rewritten, retaining its accuracy, for the case when $Q_r$ is not measured. This equation has the form

$$T_s = [Q_i \cdot (1.0 - A) + Q_a - c + (0.00036 \cdot P_a \cdot T_a + e_a - a) \cdot 1719.5 \cdot f(u) + 260 \cdot \rho_s'^{3/2} + Q_w'] / [d + (0.00036 \cdot P_a + b) \cdot 1719.5 \cdot f(u) + 8.1 \cdot \rho_s'^{3/2}] \qquad (21)$$

where $c$ and $d$ have the values

$$20°\text{F} \leq T_s \leq 32°\text{F} \quad c = 246. \quad d = 2.54$$

$$T_s < 20°\text{F} \quad c = 249. \quad d = 2.31$$

The third situation, that of $Q_c$ not equal to zero or equation 12, cannot be solved directly. However, a reasonable approximation of the energy balance, described later in the paper, can be made in this case. The basis of this approximation is the solution of equation 2 for a snow pack with no heat storage or melt; thus $Q_c$ and $Q_i' = 0$. This equation has exactly the same forms as equations 20 and 21, except that the terms that come from equation 12 involving $\rho_s'$ are eliminated. Thus, if net radiation is not known, this equation has the form

$$T_s = [Q_i \cdot (1.0 - A) + Q_a - c + (0.00036 \cdot P_a \cdot T_a + e_a - a) \cdot 1719.5 \cdot f(u)]/[d + (0.00036 \cdot P_a + b) \cdot 1719.5 \cdot f(u)] \quad (22)$$

Once $Q_m'$ and $Q_c$ are determined, the change in heat storage of the snow pack can be computed, if it is assumed that the heat transfer across the snow-soil interface does not affect heat storage. In general this is true, as the bottom layer of snow is at 32°F during most of the time snow is on the ground. In this situation heat transfer from the soil results in melt at the interface but does not affect heat storage. Even when the assumption is not valid, the amount of heat transferred at this interface is small compared with that at the air-snow interface. Thus, by ignoring $Q_s$, and since $Q_w' = Q_w$ and $Q_m' = -Q_i'$, the change in heat storage of the snow pack can be expressed as

$$Q_\theta = (Q_i + Q_m') - Q_c \quad (23)$$

by combining equations 1 and 2. The term $Q_i + Q_m'$ can be determined by realizing that $Q_m'$ plus any rain that has occurred is the water available for freezing. It is assumed that all this water will freeze within the pack until the heat storage is zero, i.e., an isothermal snow pack at 32°F. When the heat storage becomes zero, additional melt or rain water percolating down from the surface can increase the liquid-water content or, if the liquid-water capacity is satisfied, i.e., the snow is 'ripe,' it becomes runoff from the snow pack.

## TESTING THE ENERGY BALANCE EQUATIONS AT A POINT

It is difficult to obtain adequate meteorological and snow pack measurements to evaluate fully the reliability of the derived energy balance equations of the previous section. The only data found of adequate quality were those collected in the Lower Meadow lysimeter studies at the Central Sierra Snow Laboratory during the Cooperative Snow Investigations of the Corps of Engineers and the Weather Bureau. These data were collected during the active melt season in late April and early May of 1954. A complete description of the lysimeter site and observations made is contained in the *Snow Investigations Research Note No. 25* [1955].

### *Data Available*

Following is a brief description of the measurements used to evaluate the reliability of the energy balance equations that need special mention.

*Radiation.* The daytime observations of incident total hemispherical radiation were quite erratic, indicating that the radiometer measurements were not reasonable during that period. Erratic results similar to these were noted on the same type of radiometers during the *Lake Hefner Studies* [1954]. At Lake Hefner daytime values of incident terrestrial radiation were determined by a linear interpolation between values measured at night. This assumption resulted in a satisfactory energy balance at Lake Hefner. The same method was assumed to be valid at the lysimeter site.

*Wind function.* A wind function $f(u)$ of 0.0061 $u$ (in inches per inch of mercury) was determined during the snow investigations analysis of the lysimeter data. The wind velocity $u$ is in miles traveled. During the snow investigations analysis the convective heat transfer was determined as a residual of the energy and water balances. From this a value for $f(u)'$ was determined. This value is 0.023 $u$ calories $°\text{F}^{-1}$. A value of $f(u)'$ can also be determined from $f(u)$ and the Bowen ratio, assuming that $K_H$ and $K_W$ are equal. This value of $f(u)'$ in the same units is 0.079 $u$.

*Snow pack runoff.* During the snow investigations study lysimeter runoff was checked by snow tube-depth probe measurements. These measurements exceeded lysimeter runoff by 9%; thus lysimeter runoff was increased by that amount. However, a subsequent study by the Soil Conservation Service and *Cold Regions Research and Engineering Laboratory* [1965] reported that the federal snow tube overreads on the average by 8%. Based on this consideration, the original measured lysimeter runoff is used in this paper.

*Testing*

For purposes of testing the energy balance equations developed in the last section, the day was divided into two 12-hour periods. Equations 16 and 17 were used to compute snowmelt from 6 a.m. to 6 p.m., since average daytime snow surface temperature was observed to be essentially 32°F, and equation 21 was used to compute the average snow surface temperature during the night.

*Results*

Figure 1 shows a comparison of the calculated and observed snow pack runoff. The results are very encouraging. The comparison of calculated and observed average nighttime snow surface temperature, as shown in Figure 2, is also good.

*Conclusions*

From these comparisons it seems evident that it is possible to calculate snow pack runoff at a point with reasonable accuracy during the active melt season, when the necessary meteorological variables are observed. It also seems that equation 21 provides a reasonable estimate of the surface snow temperature at night under the same conditions. In addition, based on these comparisons and the general success reported on the use of the Bowen ratio, assuming $K_H$ and $K_W$ to be equal, it seems evident that the $f(u)'$ determined during the snow investigations analysis of the lysimeter data is not realistic. This should have been partially expected, since $Q_h$ in

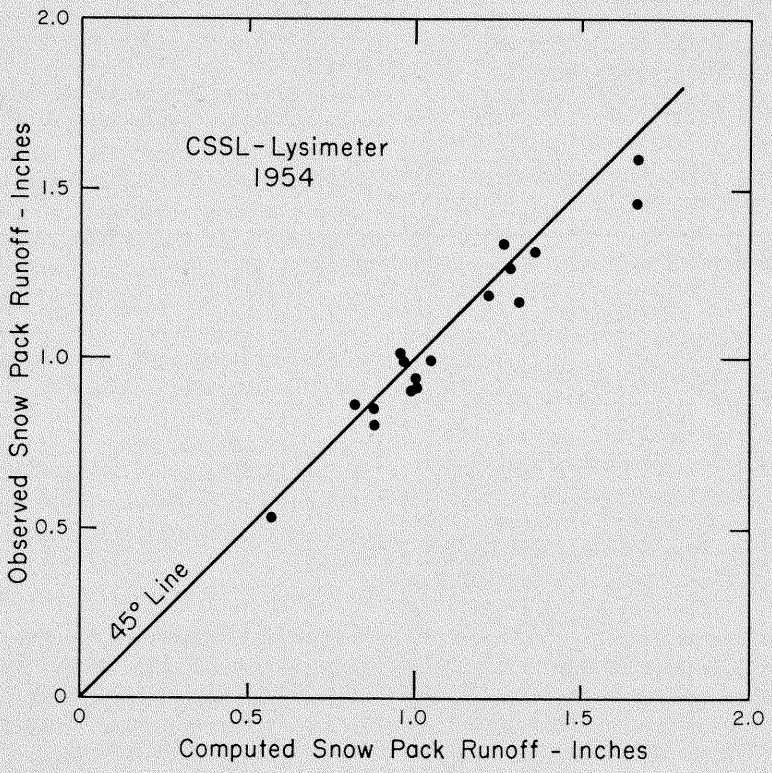


Fig. 1. Comparison of computed and observed snow pack runoff.

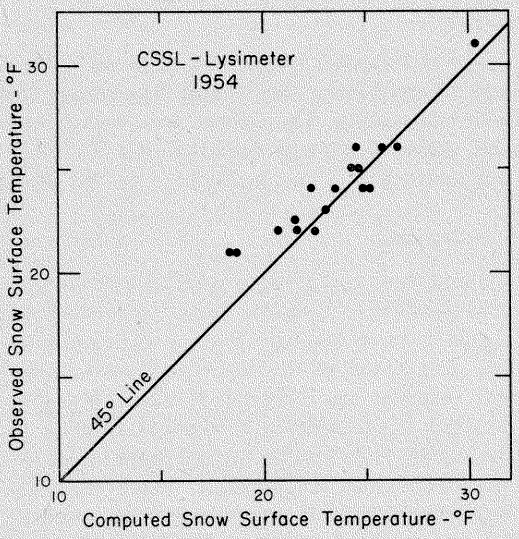


Fig. 2. Comparison of computed and observed nighttime snow surface temperature.

that analysis was determined as the relatively small difference between two large quantities, both subject to significant error.

## MODEL FOR THE SIMULATION OF WATERSHED RUNOFF

To apply the energy balance equations on a continuous basis over a watershed, certain assumptions and modifications are necessary. This section describes these and the other components of the simulation model. A flow chart of the over-all model is shown in Figure 3.

### *Modification of the Energy Balance Equations for use on a Continuous Basis over a Watershed*

As mentioned previously, an approximation is necessary to solve equation 2 when the heat storage of the snow pack is not zero at the beginning of the period. Because the vast majority of the heat transferred to or from the surface layer is with the atmosphere, equation 22 provides a basis for this estimate. The empirical technique used varies as follows, depending on whether $T_s$ as calculated by equation 22 is greater or less than 32°F.

1. If $T_s$ is calculated to be greater than 32°F, this means in general that the actual $T_s$ was less than 32°F for a portion of the period and equal to 32°F for the remainder. The difference between $T_s$ for the previous period and 32°F is a measure of that portion of the period that $T_s$ was less than 32°F, and the value of $T_s$ from equation 22 minus 32°F is a measure of the portion when $T_s$ equaled 32°F. Thus an average $T_s$ for the period can be estimated.

2. A $T_s$ of less than or equal to 32°F would be the snow surface temperature for the period if conductive heat transfer within the snow was zero. However, conductive transfer does occur in this situation, but, because of the low thermal conductivity of snow, the effect on the surface temperature is small. In this situation, it is assumed that the average $T_s$ for the period is 90% of the difference between $T_s$ from equation 22 and $T_s$ for the previous period.

Having estimated the average $T_s$ for the period, the gain or loss of heat from the snow pack can be computed.

The next step is to attempt to modify observed point data so that the data are representative over the entire watershed. The two most important modifications are the adjustment of atmospheric radiation data for the effect of forest cover and the modification of the wind function to account for the difference between wind over the watershed and that at a point.

The effect of forest cover on radiative transfer is estimated by a parameter called the effective forest cover. The watershed is divided into two imaginary parts, one completely open and the other completely forested. In the open area the appropriate incident radiation is the sum of incoming solar radiation and the terrestrial radiation from the atmosphere. In the forested portion it is assumed that there is no penetration of solar radiation through the forest canopy and that the forest is radiating to the snow as a blackbody at the temperature of the air. Intuitively, the effective forest cover should approximately correspond to the per cent forest cover multiplied by the average cover density.

When considering a watershed, equation 3 becomes

$$E = f(u_w) \cdot f(u) \cdot [(e_a + C_e) - e_s] \qquad (24)$$

where $f(u_w)$ is the wind function relating wind over the watershed to wind at a point, and $C_e$ corrects the point vapor pressure measurement to the average watershed elevation. (It should be pointed out that throughout this paper the net energy balance, based on average conditions

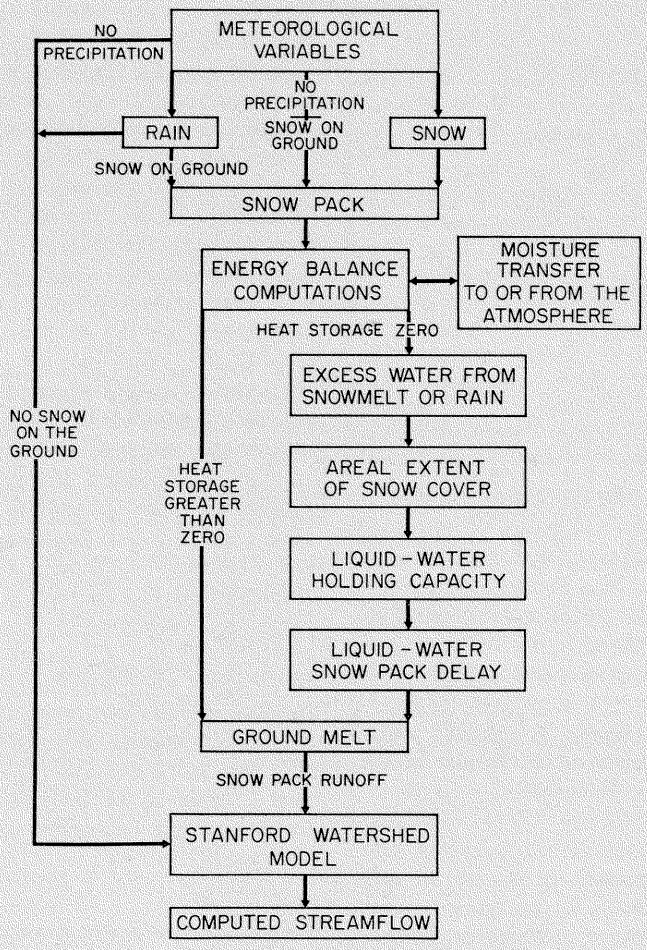


Fig. 3. Flow chart of simulation model.

over the watershed, is computed even though in nature portions of the watershed snow pack can be gaining heat while simultaneously other portions are losing heat.) The same simple form of the wind function used for the lysimeter was used for $f(u)$ and $f(u_w)$ in this case. Thus they can be expressed as

$$f(u) = K'' \cdot u \tag{25}$$

$$f(u_w) = K' \cdot f(u) \tag{26}$$

where $K'$ and $K''$ are empirical constants. For computational purposes the two functions can be combined so that the moisture transfer for the basin becomes

$$E = K \cdot u[(e_a + C_e) - e_s] \tag{27}$$

where $K = K' \cdot K''$.

*Components of the Watershed Runoff Simulation Model Involving the Snow Pack*

1. *Type of precipitation.* The form of the precipitation was estimated from the surface wet-bulb temperature. Wet-bulb temperature was determined from air and dew-point temperature by a simple empirical relationship that has good accuracy over the range of temperatures and humidities normally experienced in the United States. This relationship can be expressed as

$$T_w = T_a - (T_a - T_d) \cdot (0.12 + 0.008 \cdot T_a) \tag{28}$$

where $T_d$ is the dew-point temperature, and all temperatures are in °F. A wet-bulb temperature

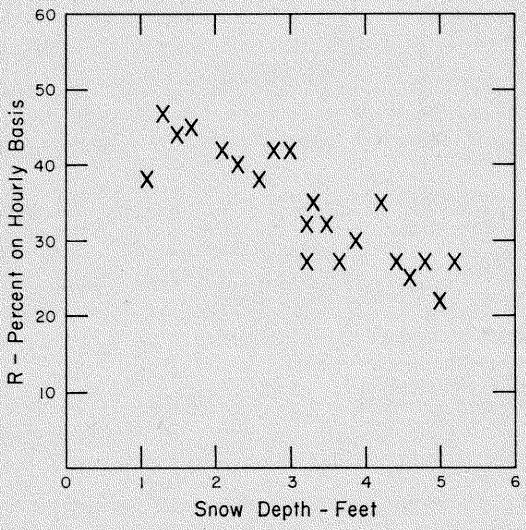


Fig. 4. Coefficient for routing melt water through ripe snow pack.

of 33°F has given the best results as the dividing point between snow and rain.

2. *Melt at the ground-snow interface.* As mentioned previously, the heat transfer at this interface is generally small and is usually toward the snow, so that melt occurs. A melt of 0.01 inch per day is used in the model.

3. *Retention of liquid-water in the snow pack.* Like soil, snow can retain liquid water by capillary tension. The capacity of the soil or snow for retaining liquid water against gravity drainage varies with physical characteristics. However, in contrast to soil, snow is in a continual state of metamorphosis; thus its liquid-water retention capacity is also changing. *Gerdel* [1954] reported that measurements on a 'ripe' snow pack in the spring in the Central Sierra showed retention of liquid water to vary from 0.7 to 5.5% of snow water equivalent, with the majority of measurements less than 2.0%. During periods when the snow pack is not 'ripe,' observations of liquid-water capacity and examination of hydrographs produced by rain on snow indicate that the retention capacity is greater than when the snow is 'ripe' and is quite variable. There have been some attempts to relate this variation to the density of the snow. However, this has not been satisfactory in general, thus suggesting that other physical characteristics of the snow have an equal or greater influence than density on the magnitude of the retention capacity. Until more is learned about the relationship between liquid-water retention capacity and the physical characteristics of the snow, physically valid estimates of retention capacity under all conditions cannot be made. Because of this lack of knowledge, a retention capacity of 2% is used for all snow pack conditions in the model. This will undoubtedly result in errors when the snow pack is not 'ripe,' especially during cases of heavy rain on fresh snow.

4. *Transmission of liquid water through snow.* Information on the movement of liquid water through snow with varying physical characteristics under various application rates is inconclusive. As reported by *Gerdel* [1954], it seems evident that under natural conditions liquid water moves downward through the snow more rapidly the closer the snow is to 'ripeness.' With fresh snow the time from application until the water reaches the ground is affected not only by the transmission rate but also by its retention capacity. This is true because the retention capacity seems to vary considerably over a relatively short time because of the rapid metamorphic changes that occur. It also seems logical to expect transmission rates to increase as application rates increase. This may happen in part because of an increase in channelization within the snow during high application rates.

Using the tabulations of surface layer snow-

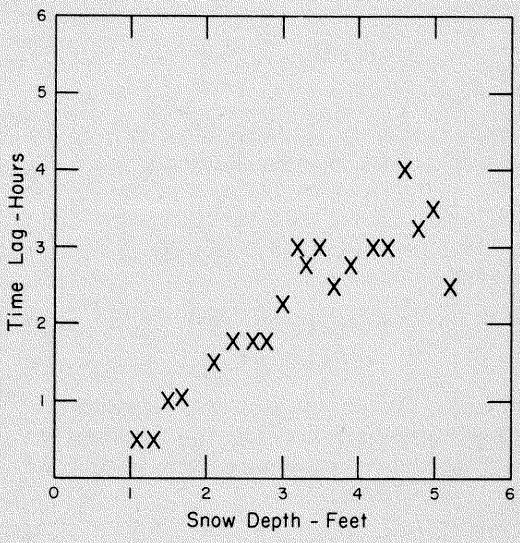


Fig. 5. Time lag for routing melt water through ripe snow pack.

melt and snow pack runoff on an hourly basis from the snow investigations analysis of the lysimeter data described in the previous section, coefficients for a simple method of routing melt water through a 'ripe' snow pack were determined. The method assumes that a per cent ($R$) of the water in transit reaches the ground during each hour. Values of $R$ and the time lag necessary to obtain a reasonable reproduction of the measured runoff are plotted against the depth of snow in Figures 4 and 5.

In the model the delay caused by the transmission of liquid water through the snow pack is based on these figures. No adjustments are made for varying application rates or for the situation when the snow is not 'ripe,' because of the limited knowledge of the physical processes involved. This, in addition to the use of a 2% retention capacity, will result in errors during periods of melt or especially during periods of heavy rain when the snow pack is fresh.

5. *Areal extent of snow cover.* The best way to determine the areal extent of snow cover is to take photographs or to make visual observations of the snow field; however, this is expensive and/or time-consuming. Thus a method to interpolate between observations or to estimate the area of cover when no observations are made is desirable.

The areal depletion curve for the watershed is used to estimate the area of cover in the model. The shape of such a curve can be determined from past data or can be estimated. For most heterogeneous watersheds the shape of this curve is very similar. When the watershed snow pack water equivalent is greater than an index value ($S_i$), 100% cover is assumed. When the water equivalent is less than $S_i$, the depletion curve of water equivalent versus per cent areal cover is used. If new snow occurs during a period of partial cover, the watershed reverts back to complete cover, at least for a short period. In this case the watershed is assumed to have 100% cover while the water equivalent is greater than $S_b + W \cdot S$, where $S_b$ is the water equivalent before the new snow ($S$) occurred, and $W$ is the portion of $S$ that must melt before partial cover again exists. When the water equivalent is between $S_b$ and $S_b + W \cdot S$, the areal cover is assumed to vary linearly between 100% and its value before the new snowfall. A diagram of this procedure is shown in Figure 6.

*Stanford watershed model.* This model, as developed by *Crawford and Linsley* [1962], [1963], [1966], at Stanford University is used to determine the time delay from when water leaves the snow pack or rain reaches the ground, to the streamflow gaging station. The Mark 3 version of the Stanford model using 6-hour time increments for moisture accounting is used. This

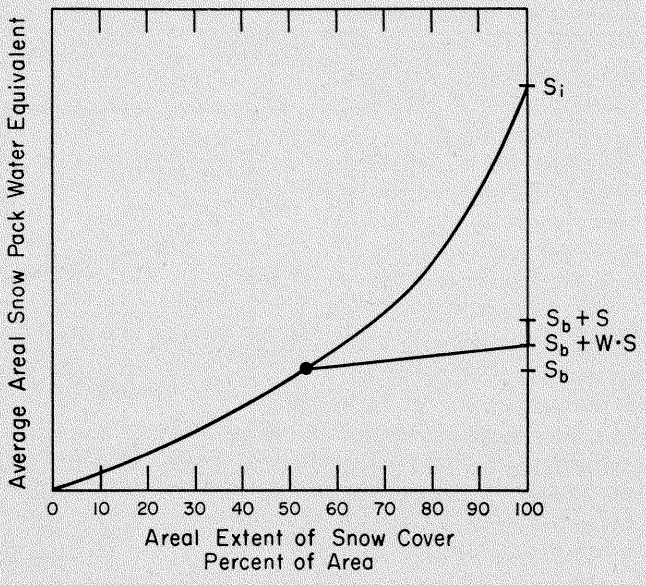


Fig. 6. Areal depletion curve for watershed snow cover.

version is basically the same as the Stanford Watershed Model 4 described by *Crawford and Linsley* [1966]. Two differences that should be mentioned are: (1) the curves for the infiltration index, interflow index, upper zone retention, and ground water inflow plus the nominal capacity of the upper zone have more controlling parameters, thus more degrees of freedom in the Mark 3 version; and (2) the overland flow routing in the Mark 3 version uses the simple relationship that a constant percentage of the water in overland detention reaches the channel during each time period, whereas the remainder is available for infiltration during the succeeding period.

Potential evapotranspiration was calculated by a combination aerodynamic-Bowen ratio approach as suggested by *Kohler and Parmele* [1967]. This is used to calculate the evapotranspiration from the soil and vegetation, as the loss of moisture by the snow to the air is computed separately. When snow is on the ground, the actual evapotranspiration is reduced proportionately to the effective forest cover and the areal extent of the snow cover.

*Testing of the model.* The most complete set of observations for testing the model is that collected during the Cooperative Snow Investigations at the Central Sierra Snow Laboratory (CSSL) near Donner Summit, California. The meteorological variables necessary, plus additional observations needed for a proper evaluation, were available for five water years, 1947–1951. A complete description of the CSSL area is contained in the summary report of the *Snow Investigations, Snow Hydrology* [1956].

*Data available.* Following are a few brief but pertinent remarks about the input data. Energy balance computations were made using the same 12-hour periods as used for the lysimeter data. A complete compilation of all measurements made at CSSL is contained in the Hydrometerological Logs of the watershed.

Reflected solar radiation was observed at the headquarters station during portions of each year. However, because the records of reflected solar radiation were not continuous and the site was void of snow before other portions of the area, watershed albedo was estimated from an empirical relationship. The form of this relationship can be expressed as

$$\begin{aligned} A &= m \qquad \text{when} \quad A_i = 0 \\ A &= m \cdot n^{A_i{}^g} \quad \text{when} \quad A_i > 0 \end{aligned} \tag{29}$$

where $m$, $n$, and $g$ are empirical constants, and $A_i$ is the albedo index that increases during melt and/or rain and decreases during new snowfall. A typical albedo curve from the relationship is shown in Figure 7. This relationship seems to work quite adequately, except during periods of new snowfall late in the melt period. During this situation the actual albedo increases rapidly and then shortly returns to its previous value. The relationship of equation 28 does not have such a shape when $A_i$ is large, as would be the case late in the melt season. Thus the model should compute too much snowmelt immediately following such an occurrence.

Observations of terrestrial radiation from the atmosphere were made only during the lysimeter studies. Thus an estimate was needed. The estimate comes from the relationship, suggested by *Anderson and Baker* [1967], which utilizes surface air temperature, vapor pressure, and incident solar radiation. It was assumed that half the estimated radiation occurred during the daytime and the other half at night.

*Method of testing.* The parameters for the Stanford Watershed Model are unchanged from those determined in previous modeling studies of the CSSL watershed. These studies used air temperature as the sole index to the energy balance. Many of the other parameters were also unchanged as appropriate values were deter-

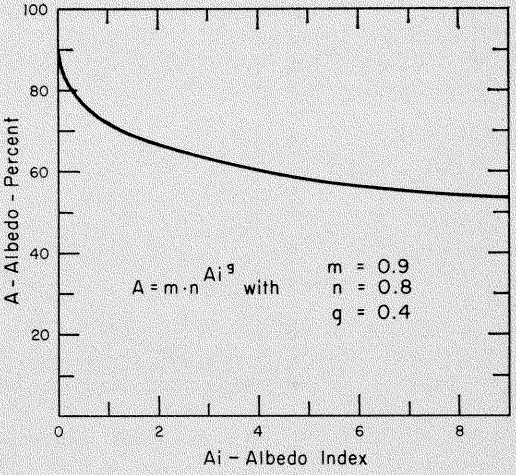


Fig. 7. Watershed albedo.

mined during separate studies, such as those of the energy balance at a point and the transmission of liquid water, previously described. The remaining parameters were determined by trial and error. The final values were based on a subjective best fit of observed versus computed mean daily flows and areal snow cover.

*Results.* In a model of this complexity, because of the assumptions and estimations involved, it is impossible to obtain unique values for all parameters. The majority of the final model parameters for the CSSL are believed to be reasonably unique. The belief that they are reasonably unique is based on the limited test-

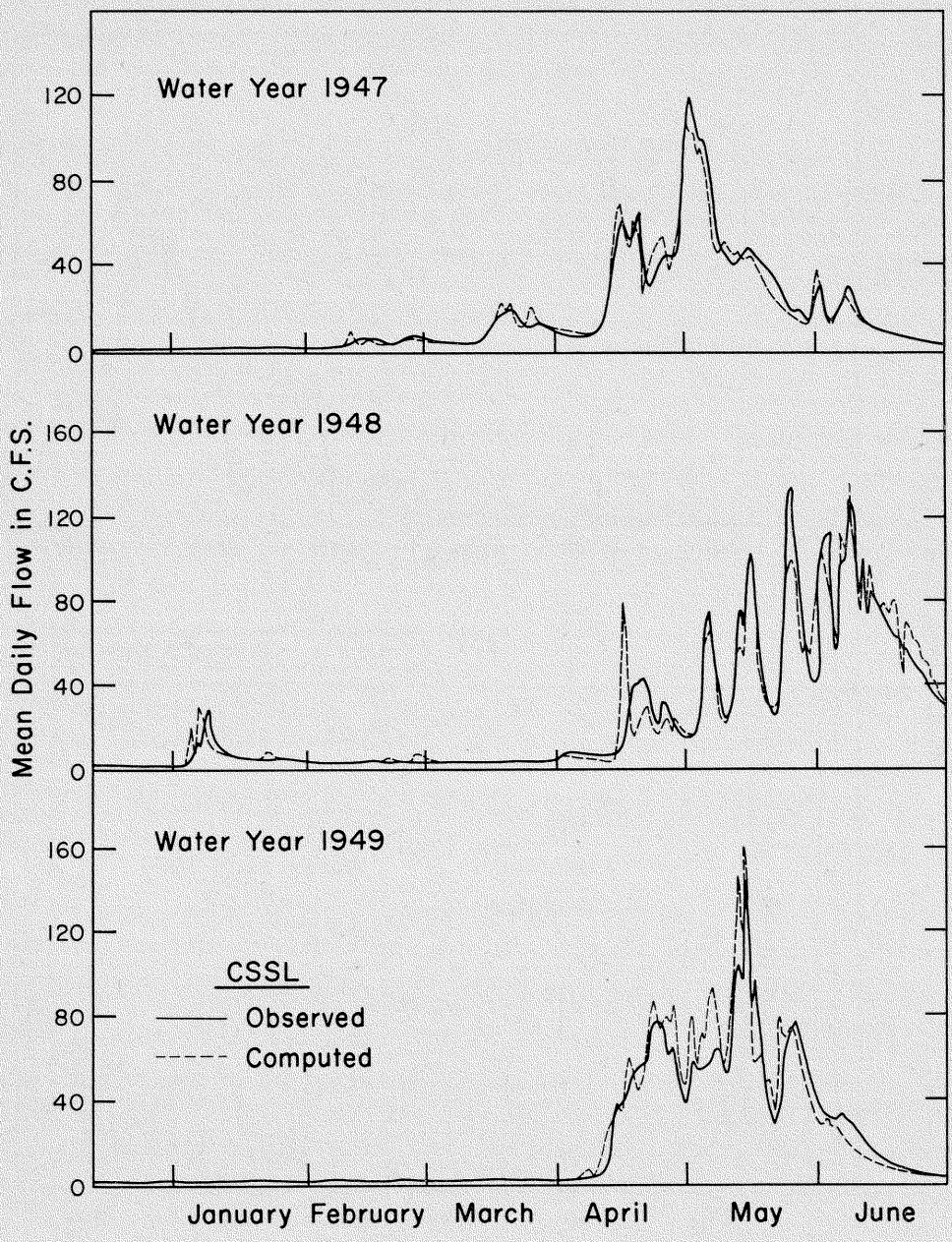


Fig. 8. Comparison of computed and observed streamflow—water years 1947–1949.

ing of the model and the much more comprehensive investigations of many of the components.

There are two components of the watershed energy balance for which reasonably unique parameters cannot be determined at present. These components are sensible and solar radiation heat transfer. The wind function and snow pack albedo for the watershed are impossible to determine accurately without additional observations of these elements throughout the area. In addition, there is a relatively high intercorrelation, especially during the melt season, at CSSL between air temperature and solar radiation. The correlation coefficient for these variables

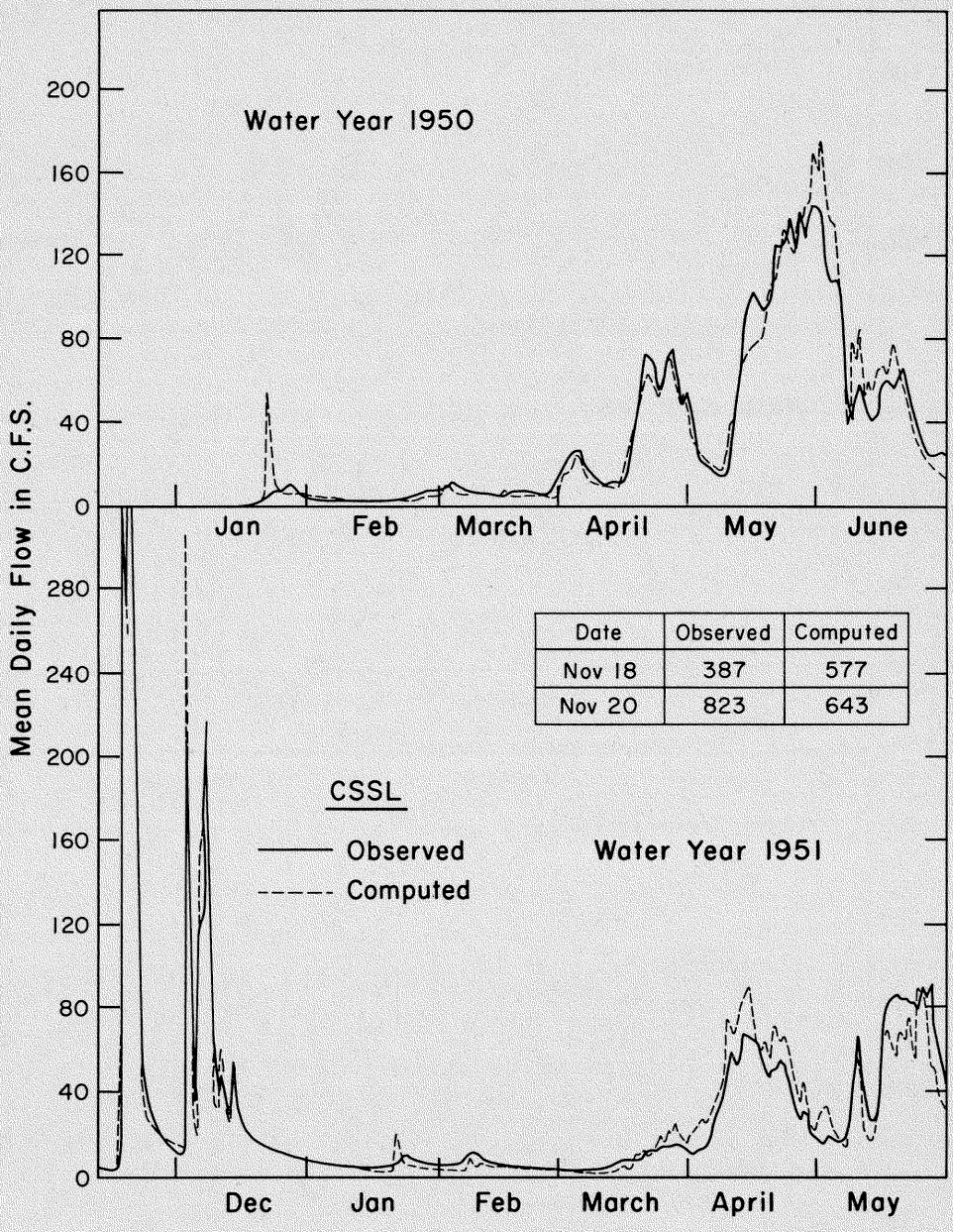


Fig. 9. Comparison of computed and observed streamflow—water years 1950 and 1951.

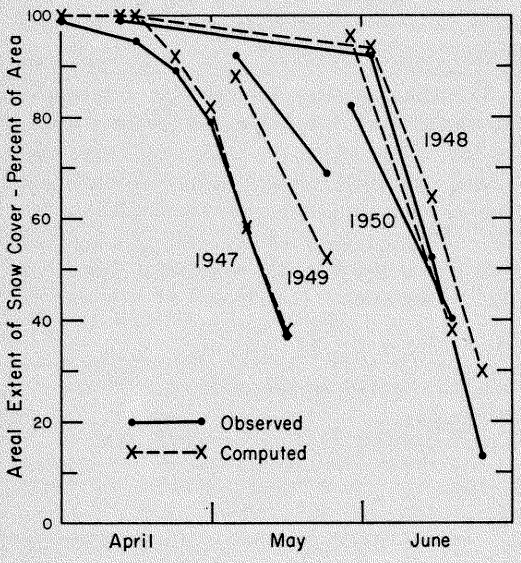


Fig. 10. Comparison of computed and observed areal snow cover.

during the spring at CSSL is in the order of 0.75. Because of these conditions the relative uniqueness of the best fit parameters for the two components was tested by varying $K$ in the wind function term and attempting to compensate for its effect on streamflow by changing $m$, $n$, and $q$ in the albedo term. Varying $K$ by $\pm 25\%$ resulted in solutions which are judged to be nearly equal to the original best fit solution. Variations in $K$ of $\pm 50\%$, however, could not be compensated for in the albedo term.

Figures 8 and 9 show the comparison between actual and computed streamflow for the original best fit parameters. In general, the reproduction is good. The runoff events of January 5 and 8, 1948; April 16, 1948; January 21, 1950; and January 21, 1951, are cases of heavy rain on fresh snow.

Figure 10 shows observed and computed areal snow cover for the same set of parameters. The procedure discussed in the previous section was

TABLE 1. Energy Balance Components

| | Per Cent Variation in $K$ from Original Best Fit | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | −25 | 0 | +25 | −25 | 0 | +25 | −25 | 0 | +25 |
| Year | Net Radiative Heat Transfer, langleys | | | Convective Heat Transfer, langleys | | | Moisture Transfer to or from the Atmosphere, inches | | |
| 1947 | 3217 | 1814 | 588 | 5850 | 7364 | 8750 | 0.16 | 0.11 | 0.01 |
| 1948 | 4835 | 3433 | 2159 | 5844 | 7365 | 8761 | 0.23 | 0.21 | 0.17 |
| 1949 | 4140 | 2898 | 1813 | 5788 | 7214 | 8723 | −0.06 | −0.17 | −0.31 |
| 1950 | 7227 | 5439 | 3804 | 6832 | 8657 | 10330 | 0.34 | 0.37 | 0.37 |
| 1951 | 5118 | 3876 | 2744 | 5317 | 6721 | 8033 | 0.00 | −0.04 | −0.10 |

Daytime (6 a.m. to 6 p.m.) only

| Date | Net Radiative Heat Transfer, langleys | | | Convective Heat Transfer, langleys | | | Condensation Heat Transfer, langleys | | |
|---|---|---|---|---|---|---|---|---|---|
| 1947 Apr. | 2439 | 2193 | 1962 | 612 | 806 | 996 | 34 | 45 | 55 |
| May | 2675 | 2191 | 1800 | 620 | 743 | 831 | 189 | 225 | 254 |
| 1948 May | 2759 | 2547 | 2324 | 666 | 896 | 1124 | 129 | 177 | 226 |
| June | 3781 | 3312 | 2814 | 617 | 774 | 891 | 224 | 281 | 324 |
| 1949 Apr. | 3368 | 3092 | 2809 | 874 | 1169 | 1463 | −89 | −116 | −141 |
| May | 3579 | 3280 | 3033 | 568 | 745 | 933 | 0 | −1 | 0 |
| 1950 May | 4016 | 3693 | 3370 | 1040 | 1377 | 1710 | 255 | 334 | 415 |
| June | 3631 | 2999 | 2487 | 634 | 745 | 835 | 251 | 293 | 327 |
| 1951 Apr. | 3094 | 2815 | 2515 | 568 | 755 | 936 | 52 | 72 | 29 |
| May | 2318 | 2083 | 1852 | 692 | 904 | 1070 | 172 | 228 | 247 |

\+ Gain in heat or moisture by the surface.
− Loss in heat or moisture by the surface.

used to compute snow cover, even though observations of this variable were available. The observations are used only to evaluate the results.

Table 1 gives the computed components of the energy balance of the snow pack on an annual basis for the original best fit parameters and the parameter sets containing the $\pm 25\%$ variation in $K$. Also shown are the daytime values of these components for the two months of highest melt each year. This shows the energy transfer components during active melt. The reason for convective transfer's being greater on an annual basis is that it is almost always positive, whereas radiative transfer is usually negative at night and during much of the accumulation period. It should also be pointed out that over the complete snow season the moisture transfer to or from the snow pack is negligible, considering the total amount of water involved.

Since the necessary data were available, the degree-day (above 32°F) melt factors that would reproduce the computed hydrograph were calculated. Melt factors were computed for surface layer melt, snow pack runoff, and streamflow runoff. Three examples of the seasonal variation in these factors are shown. Figure 11*a* shows melt factors for the surface layer assuming complete snow cover. Figure 11*b* shows melt factors for snow pack runoff. The factors in Figure 11*a* would have to be adjusted for storage within the snow pack and for actual snow cover to obtain runoff. Most of the extreme points on these plots are on days when the amount was small; thus, even though the scatter is significant, it is not as great as it seems at first glance. Though the points are not identified as to year, it should be mentioned that whereas in Figure 11*a* best fit lines for each year nearly coincide, in Figure 11*b* they do not, especially later in the melt season when the amount of snow cover differed from year to year. Figure 11*c* shows snow pack runoff versus degree-days for the month of May. It appears that, although the scatter is significant, the best fit relationship would be linear.

## COMPARISON OF ENERGY BALANCE AND MELT FACTOR APPROACHES

Before the model just described was developed, a model utilizing air temperature as the sole index to the energy balance was tested. This melt factor model was the same as the energy balance model, except that:

1. Mean daily air temperature degree-days above 32°F were the index to the heat gained by the surface layer. The degree-days were multiplied by a melt factor, which could vary with time of year, to get the total heat gain.
2. Minimum daily air temperature was the index to heat loss from the snow pack. The factor used to determine the amount of loss was decreased as heat storage increased.
3. Air temperature alone was used to determine the form of the precipitation.

Figure 12 shows the comparison between observed streamflow and that computed by each model.

## CONCLUSIONS AND RECOMMENDATIONS

Based on the discussion and results presented in the preceding sections, the following conclusions seem warranted:

1. The daily energy balance of a snow pack at a point during the active melt season can be calculated with good accuracy using equations 16 and 20 or 21. Some further studies are probably warranted to improve continuous energy balance computational techniques.
2. The reproduction of the observed hydrograph on the CSSL watershed using the energy balance model was good. Even though there are some uncertainties in the model, its generally good success suggests that it is, at least, a good approximation to the over-all physical process. Thus it can be used as a tool in further investigations of the snow accumulation and ablation processes over a watershed. The results of these investigations should in turn lead to improvements in the model.
3. At present, for the CSSL watershed the observed snow pack runoff hydrograph can be reproduced nearly as well with the melt factor model as with the energy balance model. Since it would seem reasonable from a knowledge of the energy balance that the denser the forest cover the better air temperature alone would be as an index to heat exchange, and since the CSSL area is relatively open, then the preceding statement may hold for most watersheds. However, it seems that more confidence can be placed on the energy balance approach during extreme

events, when in many instances the relationships between air temperature and the other variables deviate from the normal. Furthermore, the energy balance model adequately reproduced small rises during the accumulation season and the first rises of the melt season, whereas the melt factor model did not.

In addition to suggestions made previously, the following are some recommendations for possible future study in snow hydrology:

1. Few studies have been made of the retention and transmission of liquid water in a snow pack, especially snow that is not 'ripe.' Such

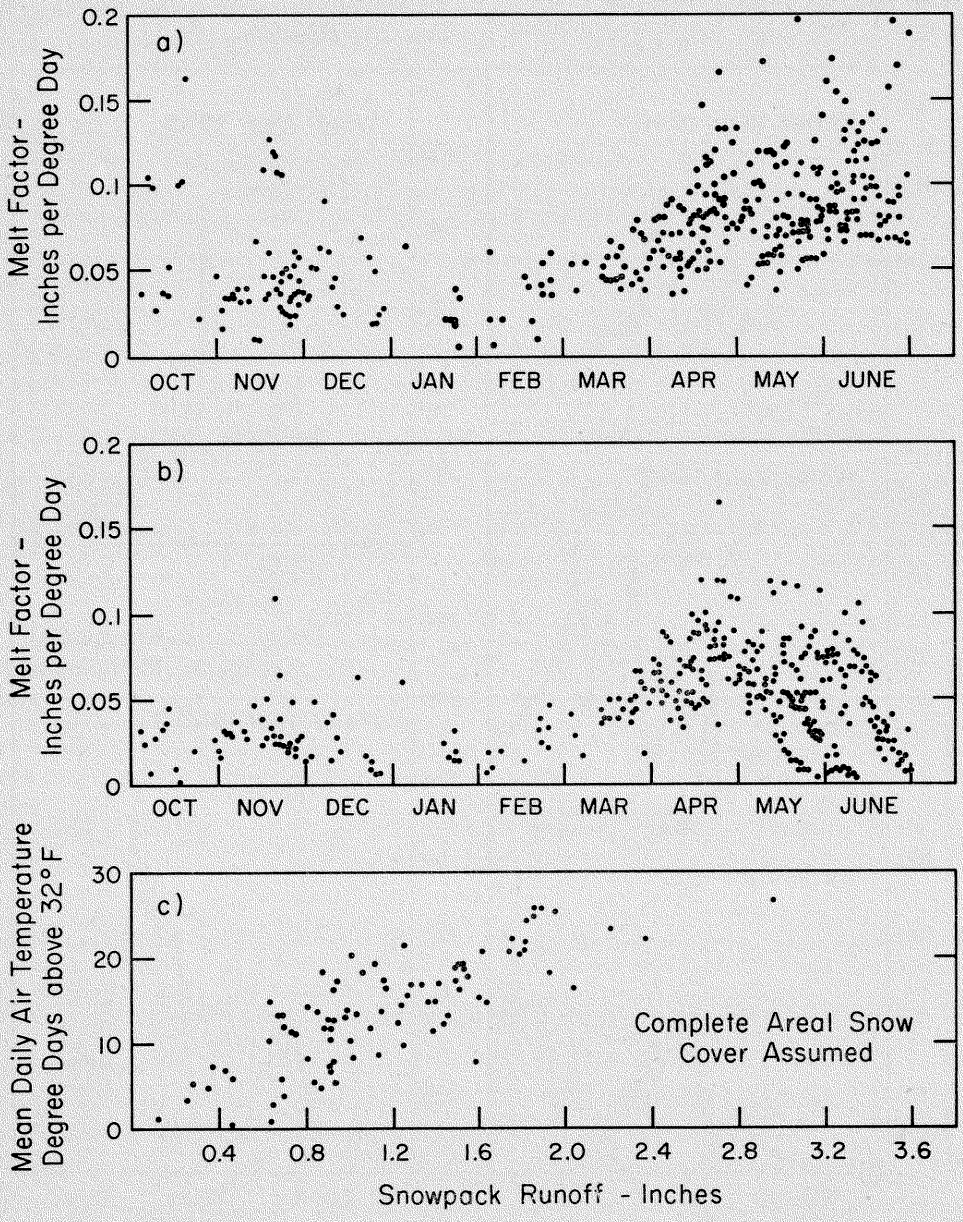


Fig. 11. Variations in melt factors for (*a*) surface layer melt with complete areal cover; (*b*) snow pack runoff; and (*c*) the month of May at CSSL; water years 1947–1951.

information is needed, especially for the simulation or forecasting of the amount and timing of runoff caused by heavy rain on fresh snow.

2. The applicability of the models, at present, is limited to areas with a relatively heavy snow pack. In addition to more testing on this type of area with varying forest configurations and elevation ranges, techniques could be developed to handle special problems that exist in other areas where snow pack runoff is important. Some of these problems are the effect of frozen ground on runoff and the effect of capillary slush layers that can occur at the snow-soil interface and affect the timing of runoff considerably in a shallow snowpack.

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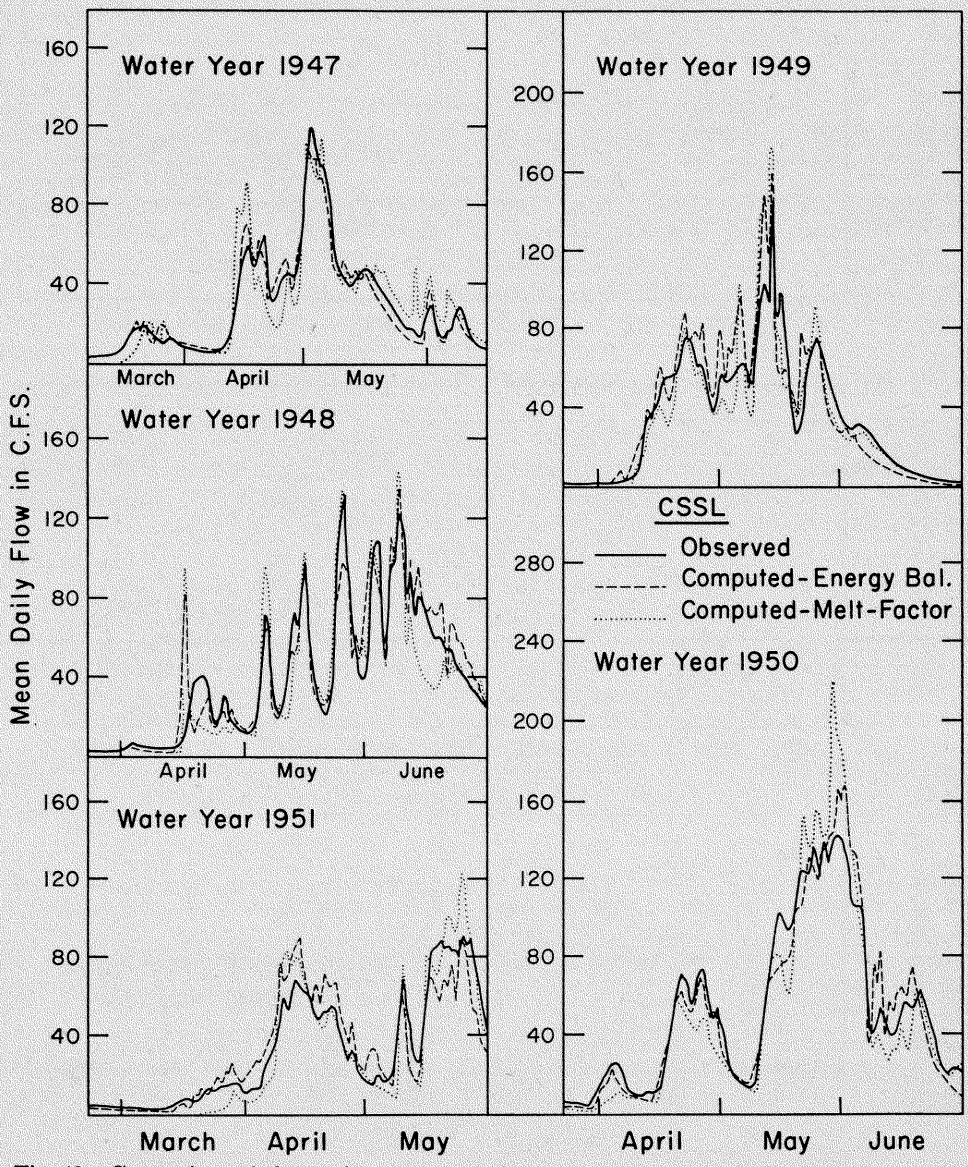


Fig. 12. Comparison of observed streamflow and streamflow computed by energy balance and melt factor models.

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