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ABSTRACT

The effects of the size of the Δt time step used in the integration of the implicit difference equations of unsteady open-channel flow are determined for numerous typical hydrographs with durations in the order of days or even weeks. Truncation errors related to size of the Δt time step cause a numerical distortion (dispersion and attenuation) of the computed transient. The magnitude of the distortion is related directly to the size of the time step, the length of channel reach, and the channel resistance and inversely to the time of rise of the hydrograph. The type of finite difference expression which replaces spatial derivatives and non-derivative terms in the partial differential equations of unsteady flow has an important influence on the magnitude of the numerical distortion, as well as the numerical stability of the implicit difference equations. Time step sizes in the range of 3 to 6 hrs generally tend to minimize the combination of required computation time and numerical distortion of transients having a time of rise of the order of several days.

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EFFECTS OF TIME STEP SIZE IN IMPLICIT DYNAMIC ROUTING[1]

D. L. Fread[2]

ABSTRACT. The effects of the size of the Δt time step used in the integration of the implicit difference equations of unsteady open-channel flow are determined for numerous typical hydrographs with durations in the order of days or even weeks. Truncation errors related to the size of the Δt time step cause a numerical distortion (dispersion and attenuation) of the computed transient. The magnitude of the distortion is related directly to the size of the time step, the length of channel reach, and the channel resistance and inversely to the time of rise of the hydrograph. The type of finite difference expression which replaces spatial derivatives and non-derivative terms in the partial differential equations of unsteady flow has an important influence on the magnitude of the numerical distortion, as well as the numerical stability of the implicit difference equations. Time step sizes in the range of 3 to 6 hrs generally tend to minimize the combination of required computation time and numerical distortion of transients having a time of rise of the order of several days.
KEY TERMS: open-channel flow, unsteady flow equations, finite differences, implicit method, truncation errors)

INTRODUCTION

Unsteady or transient flow in open channels such as rivers, canals, reservoirs, etc., may be simulated by a mathematical model based on the complete one-dimensional unsteady flow equations which conserve the mass and the momentum of the flow. Analytical solutions to these nonlinear partial differential equations do not exist. However, they may be solved by numerical techniques which use algebraic finite difference equations to approximate the partial differential equations. It is essential to utilize a digital computer to perform the numerous computations required by this solution technique.

Numerous finite difference techniques for numerically integrating the unsteady flow equations have been reported in the literature. They may be categorized into the following four methods of solution:

1. Implicit method [Abbott and Ionescu, 1967; Lai, 1967; Baltzer and Lai, 1968; Amein, 1968; Dronkers, 1969; Amein and Fang, 1970; Strelkoff, 1970; Kamphuis, 1970; Gunaratnam and Perkins, 1970; Contractor and Wiggert, 1971; Fread, 1972];
2. Explicit method [Isaacson, Stoker and Troesch, 1956; Stoker, 1957; Liggett and Woolhiser, 1967; Dronkers, 1969; Garrison, Granju and Price, 1969; Strelkoff, 1970; Strelkoff and Terzidis, 1970];

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3. Characteristic method with curvilinear net [Lister, 1960; Amein, 1966; Streeter and Wylie, 1967; Liggett and Woolhiser, 1967; Wylie, 1969; Fread and Harbaugh, 1972] ; and
4. Characteristic method with rectangular net [Lister, 1960; Streeter and Wylie, 1967; Baltzer and Lai, 1968; Mozayeny and Song, 1969; Wylie, 1970; Yevjevich and Barnes, 1970].

Of the four methods, the implicit method appears to be best suited for modeling transient flows with durations in the order of days or weeks such as the natural floods occurring in large river systems. The implicit method, unlike the other methods, theoretically does not restrict the size of time step because of the numerical stability characteristics of the finite difference equations. Large time steps can enable the implicit method to be more computationally efficient than the other methods, particularly for long duration transients.

The aim of this paper is to investigate the effect of large time steps on the accuracy of solutions obtained from the unsteady flow equations by the implicit finite difference technique for transient flows of durations in the order of days and weeks.

UNSTEADY FLOW EQUATIONS

The unsteady flow equations, the equation of continuity (conservation of mass) and the equation of motion (conservation of momentum), can be respectively expressed in the divergence form as:

$$\frac{\partial A}{\partial t} + \frac{\partial (AV)}{\partial x} - q = 0 \tag{1}$$

$$\frac{\partial (AV)}{\partial t} + \frac{\partial (AV^2)}{\partial x} + gA \left(\frac{\partial h}{\partial x} + S_f\right) - qv_x = 0 \tag{2}$$

in which the resistance slope, S_f, is given by the Manning equation, i.e.,

$$S_f = n^2 \mid V \mid V / [2.21 \left(\frac{A}{P}\right)^{4/3}] \tag{3}$$

The terms in the above equations are defined as: x = longitudinal distance along the channel, positive in the downstream direction; t = time; A = cross-sectional area of flow; V = mean velocity of flow across a section, positive in the downstream direction; h = water surface elevation; q = known lateral inflow or outflow per unit length along the channel, positive if inflow; v_x = velocity of lateral flow in the direction of the channel flow; S_f = resistance slope; n = Manning roughness coefficient; P = wetted perimeter of the flow cross section; and g = acceleration of gravity.

A derivation of the equations may be found in several references, e.g., Stoker [1957], Chow [1959] and Strelkoff [1969]. It is assumed in the derivation that the flow is one-dimensional in the sense that flow characteristics such as depth and velocity are considered to vary only in the longitudinal x-direction of the channel. It is further assumed that: 1) the velocity is constant and the water surface is horizontal across any section perpendicular to the longitudinal axis; 2) the flow is gradually varied with hydrostatic pressure prevailing at all points in the flow such that the vertical acceleration of water particles may be neglected; 3) the longitudinal axis of the channel can be approximated by a straight line; 4) the bottom slope of the channel is small; 5) the bed of the channel is fixed, i.e., no scouring or deposition is assumed to occur; 6) the resistance coefficient for steady uniform turbulent flow is considered applicable, and an empirical resistance equation such as the Manning equation describes the resistance effects; and 7) the flow is incompressible and homogeneous in density.

Equation 1 and 2 make up a system of two nonlinear, first order, first degree partial differential equations of the hyperbolic type. They have x and t as independent variables and h and V as dependent variables. The other terms are constants or are functions of independent and/or dependent variables, i.e., $A\,(x, h)$, $S_f\,(x, h, V)$, $q\,(x, t)$, $v_x\,(x, t)$, $n\,(x, h)$ and $P\,(x, h)$.

In order to obtain solutions to the unsteady flow equations, it is necessary to specify boundary and initial conditions. Boundary conditions are conditions specified at fixed values of x for various time. These values include discharge or water surface elevation versus time, or a stage-discharge relation for the upstream and downstream extremities of the channel reach. Initial conditions are conditions specified at fixed values of time at various spatial locations. An initial flow profile for the channel reach may be determined from a backwater computation [Fread and Harbaugh, 1971] and used as an initial condition. Besides boundary and initial conditions, lateral flows, channel geometry and resistance coefficients must be prescribed a priori.

IMPLICIT FINITE DIFFERENCE SOLUTION

Equations 1 and 2 may be approximated by algebraic finite difference equations; and the continuous x-t region in which solutions of h and V are desired can be represented by a rectangular net of discrete points. The net points are defined by the intersection of straight lines drawn parallel to the axes of the x-t region. Lines parallel to the x-axis are time lines and have a spacing of Δt which need not be constant. Lines parallel to the t-axis represent locations along the channel and have a spacing of Δx which need not be constant. Each discrete point may be identified by a double subscript (i, j); the first designates the x-position and the second designates the time line.

In the implicit finite difference solution, the time derivatives are approximated by a forward difference quotient centered between the i^{th} and $i + 1$ points, i.e.,

$$\frac{\partial K}{\partial t} \simeq \frac{K_{i,j+1} + K_{i+1,\,j+1} - K_{i,j} - K_{i+1,j}}{2\,\Delta t_j} \tag{4}$$

where K represents any function or variable. The spatial derivatives are approximated by a forward difference quotient positioned between two adjacent time lines according to weighting factors of θ annd $(1\text{-}\theta)$, i.e.,

$$\frac{\partial K}{\partial x} \simeq \frac{\theta\,(K_{i+1,\,j+1} - K_{i,\,j+1})}{\Delta x_i} + \frac{(1\text{-}\theta)\,(K_{i+1,\,j} - K_{i,\,j})}{\Delta x_i} \tag{5}$$

Functions other than derivatives are approximated by using weighting factors similar to Equation 5. Thus,

$$K \simeq \frac{\theta (K_{i,j+1} + K_{i+1,j+1})}{2} + \frac{(1-\theta)(K_{i,j} + K_{i+1,j})}{2} \tag{6}$$

Upon substituting the finite difference operators defined by Equations 4, 5, and 6 into the unsteady flow Equations 1 and 2, the following implicit difference equations are obtained:

$$\frac{A_{i,j+1} + A_{i+1,j+1} - A_{i,j} - A_{i+1,j}}{2\Delta t_j} + \theta \left[\frac{(AV)_{i+1,j+1} - (AV)_{i,j+1}}{\Delta x_i} - q_{i,j+1}\right]$$
$$+ (1-\theta)\left[\frac{(AV)_{i+1,j} - (AV)_{i+1,j}}{\Delta x_i} - q_{i,j}\right] = 0 \tag{7}$$

$$\frac{(AV)_{i,j+1} + (AV)_{i+1,j+1} - (AV)_{i,j} - (AV)_{i+1,j}}{2\Delta t_j} + \theta \left[\frac{(AV^2)_{i+1,j+1} - (AV^2)_{i,j+1}}{\Delta x_i}\right.$$

$$+ \frac{g(A_{i,j+1} + A_{i+1,j+1})}{2} \frac{(h_{i+1,j+1} - h_{i,j+1})}{\Delta x_i} + \frac{S_{f_{i,j+1}} + S_{f_{i+1,j+1}}}{2} - (qv_x)_{i,j+1}$$

$$+ (1-\theta)\left[\frac{(AV^2)_{i+1,j} - (AV^2)_{i,j}}{\Delta x_i} + \frac{g(A_{i,j} + A_{i+1,j})}{2} \frac{(h_{i+1,j} - h_{i,j})}{\Delta x_i} + \right.$$

$$\left.\frac{S_{f_{i,j}} + S_{f_{i+1,j}}}{2} - (qv_x)_{i,j}\right] = 0 \tag{8}$$

A weighting factor of $\theta = 1$ yields the fully implicit scheme used by Baltzer and Lai [1968]. A weighting factor of $\theta = 1/2$ produces the "box" scheme used by Amein [1968], Amein and Fang [1970], and Contractor and Wiggert [1971].

Equations 7 and 8 form a system of two algebraic equations which are nonlinear with respect to the unknowns, the values of h and V at the net points $(i, j + 1)$ and $(i + 1, j + 1)$. The terms A and S_f are known functions of h and/or V. The terms associated with the net points (i, j) and $(i + 1, j)$ are known from either the initial conditions or previous computations.

The two equations cannot be solved for the unknowns since there are two more unknowns than equations; however, by considering all N number of points along the x-axis simultaneously, a solution may be obtained. In this way, a total of $(2N - 2)$ equations with $2N$

unknowns may be formulated by applying Equations 7 and 8 recursively to the $(N - 1)$ rectangular grids along the x-axis. The boundary conditions at the upstream and downstream extremities of the channel reach provide two additional equations which are necessary for the system of equations to be sufficiently proposed to yield a solution. The resulting system of $2N$ nonlinear equations with $2N$ unknowns must be solved by an iterative procedure. A functional iterative process, called Newton-Raphson Iteration [Crandall, 1956; Amein and Fang, 1970], is used to solve the nonlinear system. The iterative process converges to a solution of acceptable accuracy at a quadratic rate; this may be improved by using parabolic extrapolation to obtain the first approximation of the solution from solutions determined at previous times. The coefficient matrix of the linearized system of equations has a banded structure which lends itself to very efficient solution algorithms, e.g., [Fread, 1970].

STABILITY OF THE IMPLICIT EQUATIONS

The solution of a system of finite difference equations requires that numerical errors of round-off, introduced in the computational procedure, not be amplified into an unlimited error. The stability of the nonlinear difference Equations 7 and 8 can be investigated by a Fourier analysis of the error propagation properties of linearized forms of the difference equations. This stability analysis, known as the von Neumann method [O'Brien, Hyman, and Kaplan, 1951] has been used by various investigators, e.g. Abbott and Ionescu [1967] and Leendertse [1967], to show that, in general, an implicit finite difference formulation of the unsteady flow equations is unconditionally stable for any ratio of $\Delta x/ \Delta t$ when the θ weighting factor is restricted to the range, $1/2 \leq \theta \leq 1$. Thus, according to this stability analysis, the stability of the implicit method does not depend on the ratio, $\Delta x/ \Delta t$, as do the explicit and characteristic methods. However, the inability to include in the stability analysis the nonlinearities of Equations 7 and 8, as well as the effects of boundary conditions, causes the von Neumann technique to be heurisitic and somewhat inconclusive. Under certain conditions, the implicit difference equations have been observed to exhibit instabilities [Liggett and Woolhiser, 1967; Baltzer and Lai, 1972]. In this investigation, numerical instabilities were encountered for certain upstream boundary hydrographs and Δt time steps; this will be discussed later

ACCURACY OF THE IMPLICIT EQUATIONS

Solutions obtained from the implicit difference Equations 7 and 8 have been mathematically shown to converge to the true solutions of the partial differential Equations 1 and 2 as Δx and Δt approach zero [Abbott and Ionescu, 1967 and Leendertse, 1967]. Thus, if channel length and the irregularity of channel geometry are used to select Δx, the accuracy of the implicit difference solution decreases as the size of the time step increases.

Truncation errors, related to the magnitude of the time step, arise during the integration of the implicit difference equations. The truncation errors distort the solution via numerical dispersion and attenuation of the computed transient. Henceforth, the truncation error in the solution will be referred to as "numerical distortion." Also, as will be shown later, the characteristics of the discharge hydrograph at the upstream extremity of the channel reach have a significant effect on the accuracy of the solution.

The characteristics of the numerical distortion are investigated herein via numerical experiments in which Equations 7 and 8 are applied to upstream boundary transients described by the following four-parameter, Pearson Type III distribution:

$$Q(t) = Q_o [1 + (p-1)(\frac{t}{\tau})^{(\frac{1}{\gamma - 1})} e^{p(\frac{1-t}{\tau})}] \tag{9}$$

in which

$$p = Q_{max} / Q_o \tag{10}$$

$$\gamma = \tau_g / \tau \tag{11}$$

The terms in the above equations are defined as follows: $Q(t)$ = discharge at any time (t), Q_o = initial steady discharge as computed by the Manning equation, Q_{max} = maximum discharge at the upstream boundary during the transient flow condition, τ = time of occurrence of Q_{max}, τ_g = time associated with the center of gravity of the upstream hydrograph, p = hydrograph amplification coefficient, and γ = a skewness coefficient of the upstream hydrograph.

The downstream boundary condition is specified by the following implicit stage-velocity relationship which is corrected for transient effects:

$$V = \frac{1.486}{n} (\frac{A}{P})^{2/3} S_f^{\;1/2} \tag{12}$$

in which

$$S_f = (qv_x - \frac{\partial (AV)}{\partial t} - \frac{\partial (AV^2)}{\partial x} - \frac{\partial h}{\partial x}) / gA \tag{13}$$

This boundary condition allows the transient to pass the downstream extremity of the channel reach with no numerical reflection.

The primary objective of this investigation is to study the effect of the size of the time step on the solutions of the implicit difference equations. Therefore, selected parameters describing the physical characteristics of the channel reach are held constant throughout the study except in special instances where a single parameter is perturbed in order to determine its effect on the results. The selected channel parameters are as follows: channel reach length (L) = 100 miles; channel bottom slope (S_o) = 1/5280 ft per ft; Manning roughness coefficient (n) = 0.03; wide rectangular cross-section with surface width (B) = 2000 ft; number (N) of Δx sub-reaches = 10, and intial depth of flow (Y_o) = 5 ft. Convergence criteria for h and V in the iterative solution were chosen as: $| h^{k+1} - h^k | \leq 1 \times 10^{-6}$ and $| V^{k+1} - V^k | \leq 1 \times 10^{-6}$, where the superscript k denotes the number of iterations.

The effect of the magnitude of the time step on the accuracy of the computed solutions is determined by systematically increasing the time step from Δt_c, a relatively small value in the order of minutes, to a relatively large value of 12 hours. The Δt_c time step is the maximum size time step that can be used in an explicit method; it is computed from the Courant condition [Stoker, 1957; Strelkoff, 1970] which insures numerical stability when friction effects are relatively small:

$$\Delta t_E \leq \Delta x_i / [\,| V_i | + (gA/B)_i^{\;1/2}] \;...\; (\text{minimum for } i = 1 \;...\; N) \tag{14}$$

The stage hydrographs obtained using Δt_c in Equations 7 and 8 are considered the standards to which the solutions computed with Δt time steps of 1, 3, 6 and 12 hours are compared.

This follows the approach used by Abbott and Ionescu [1967] for testing the effect of the magnitude of Δt in direct finite difference approximations of the unsteady flow equations. The fact that the truncation error is a minimum when the time step is Δt_c follows from a Taylor series analysis of Equations 7 and 8, as well as from the fact that as Δt increases beyond Δt_c the response properties of the computational system depart from those of the physical system.

Deviations from the standard hydrographs are measured by the following relative root mean square error (S_e) and relative error of the peak (P_e) of the hydrographs:

$$S_e = \frac{100\left[\sum_{i=1}^{i=n} (y_i - ys_i)^2\right]^{1/2}}{n^{1/2}\, ys_p} \tag{15}$$

$$P_e = 100\,(1 - y_p/ys_p) \tag{16}$$

in which n = total number of hydrograph values being compared, y_i = stage value computed with a particular Δt time step, ys_i = stage value computed with a Δt_c time step, y_p = maximum (peak) value of y_i, and ys_p = maximum value of ys_i.

Figures 1 and 2 illustrate typical numerical distortions of the computed dyrographs at the downstream boundary for two variations in the upstream boundary condition. In Figure 1, the time of rise (τ) is 48 hours, while in Figure 2, τ is 120 hours. The hydrographs obtained with a time step of 12 hours differ from those computed with a time step of 0.5 hour. The rising limb of the former occurs earlier than the latter, while the falling limb is delayed and the peak is

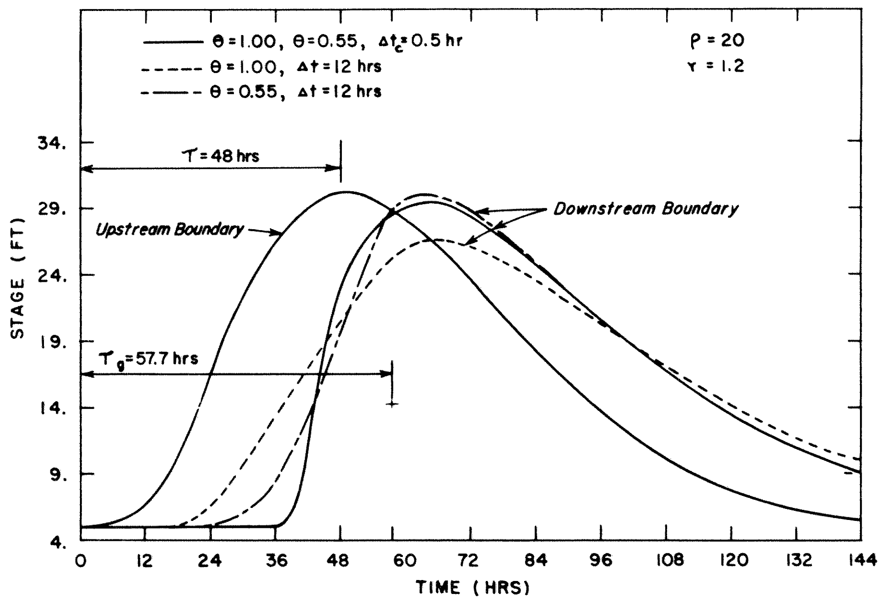

Figure 1. Distortion of computed downstream stage hydrograph for large Δt steps when θ is varied and τ = 48 hours.

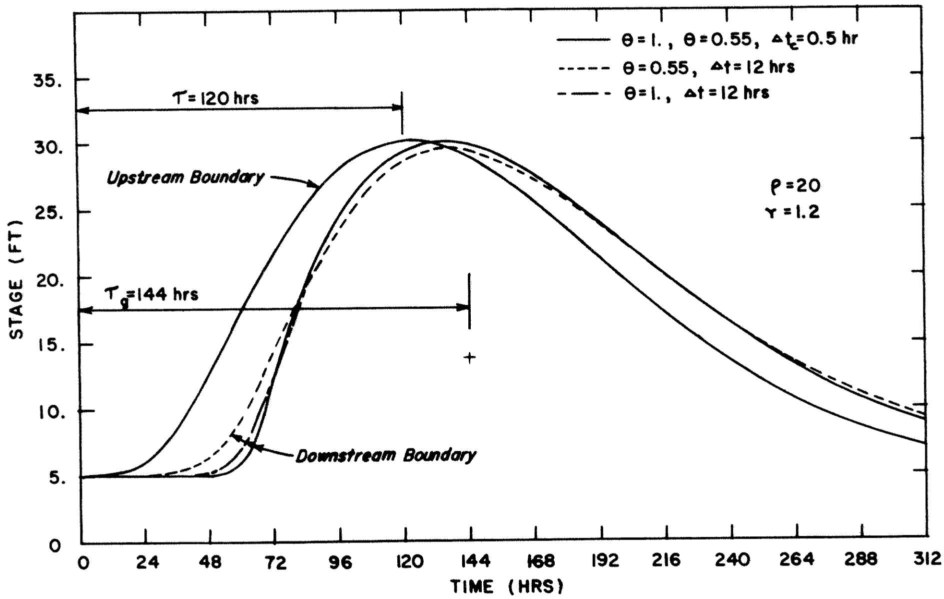

Figure 2. Distortion of computed downstream stage hydrograph for large Δt steps when θ is varied and $\tau = 120$ hours.

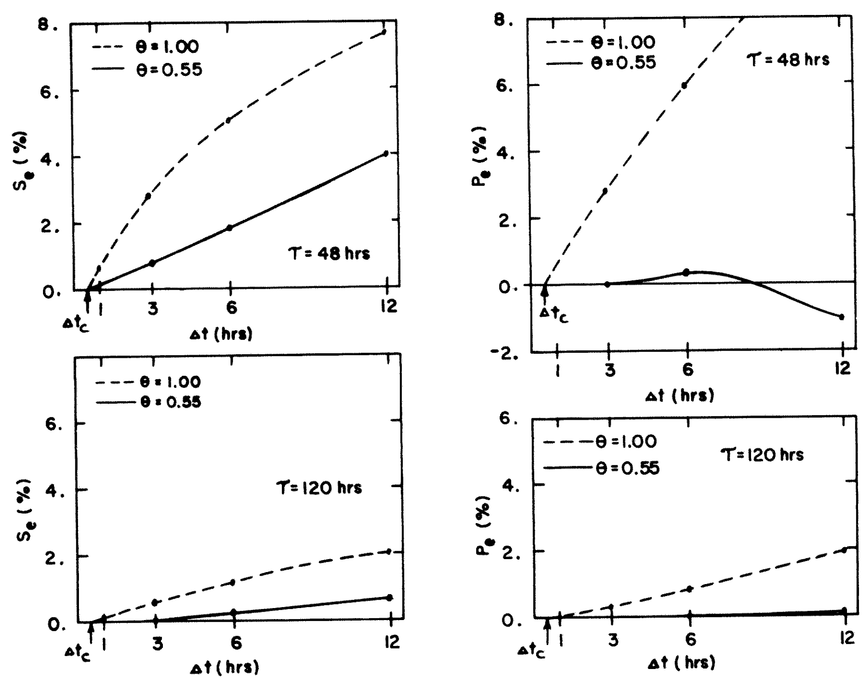

Figure 3. Effect of θ and τ on the distortion of the computed stage hydrograph at the downstream boundary for various Δt time steps having $p = 20$ and $\gamma = 1.2$.

attenuated. The distortion is more pronounced in Figure 1 than in Figure 2 for the same values of Δt and θ. Also, for a single τ value, the distortion is significantly greater for $\theta = 1$ than for $\theta = 0.55$.

A quantitiative evaluation of the numerical distortion, in terms of S_e and P_e, is shown in Figure 3. The influence of θ and τ on the degree of distortion is significant. This was also observed for other test hydrographs. Thus, it may be concluded that the lower range of allowable θ values minimizes the distortion (dispersion and attenuation) which results from the use of large time steps in the integration of the implicit difference equations. Also, the degree of distortion becomes less as the time of rise of the input hydrograph increases. Several correlations of S_e with the size of the Δt time step are shown in Figure 4. The correlations are given for various τ and p values of the upstream boundary hydrograph. The S_e error is associated with the stage hydrographs computed at the downstream boundary of the 100 mile channel reach described previously.

An examination of Figure 4 yields the following information concerning the numerical distortion resulting from the use of Δt time steps considerably larger than those determined from the Courant Condition (Equation 14):

1) The magnitude of S_e increases with the size of the Δt time step;
2) As τ, the time of rise of the upstream hydrograph increases, the slopes of the $(S_e, \Delta t)$ curves decrease;
3) The magnitude of S_e is less than 1% for $\tau \geq 96$ hours and $\Delta t \leq 12$ hours.

The solid curves in Figure 4 are applicable for a θ of 0.55, a value chosen so as to minimize numerical distortion while conservatively insuring theoretical stability of the computations. The

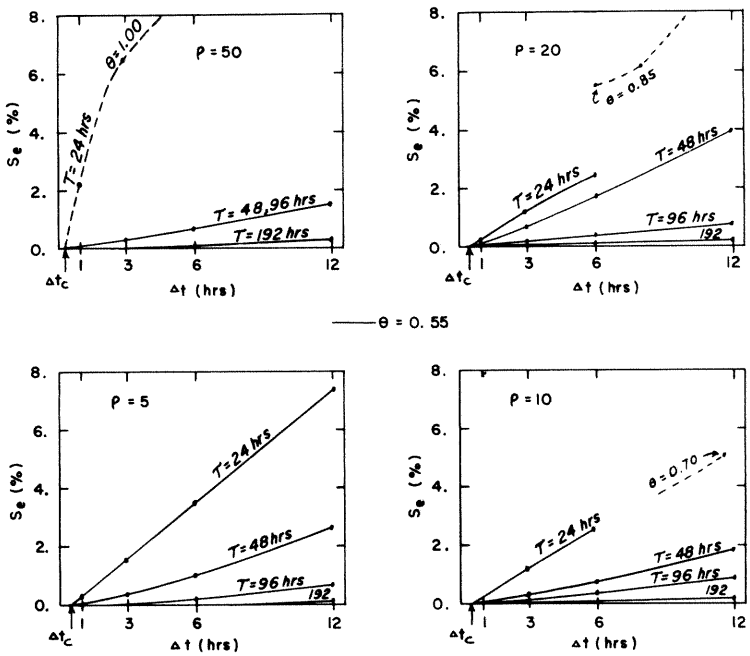

Figure 4. Correlation of S_e error (for the stage at the downstream boundary) with the Δt time step for various upstream boundary hydrographs having $\gamma = 1.2$.

dashed portion of the curves are applicable to θ values greater than 0.55 which are required for numerical stability since lesser values of θ cause instabilities to arise in the iterative solution of the nonlinear difference equations. The selected θ values are optimal in that the magnitude of numerical distortion is minimized while numerical stability is achieved. The optimal θ values vary with Δt and τ. An illustration of the variation with Δt and τ is given in Figure 5 for a p of 20. By inspecting Figures 4 and 5, it can be seen that the tendency for stable numerical computations decreases with increasing values of Δt and with decreasing values of θ and τ.

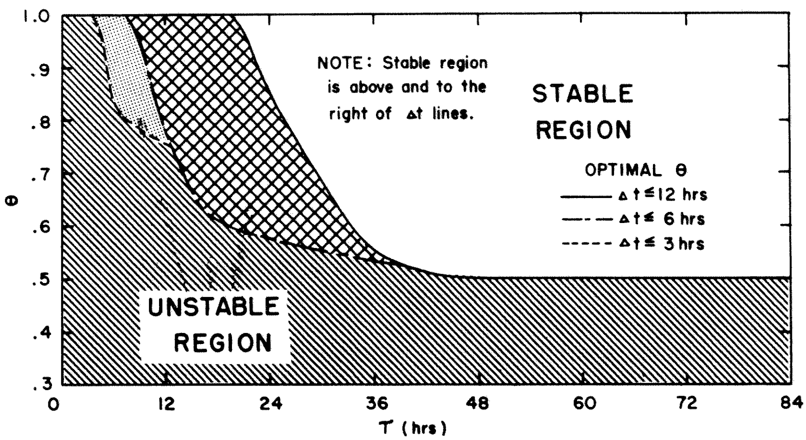

Figure 5. Optimal θ for numerical stability and minimum distortion for various τ, Δt, and p = 20, γ = 1.2.

The effect of the Δt time step size on the attenuation of the computed stage hydrographs at the downstream boundary is presented in Figure 6 for various combinations for τ and p. In Figure 6, P_e is negligible for τ values greater than 48 hours; however, P_e can be significant for $\Delta t > 3$ hour when $\tau < 48$ hours.

The results presented thus far are applicable for the constant channel parameters selected previously. In order to determine if the numerical distortion resulting from large time steps is sensitive to the values of the channel parameters, these are perturbed and the resulting effects on S_e and P_e are observed. The observed effects may be summarized by the following approximation:

$$(S'_e, P'_e) \simeq \eta . (S_e, P_e) \tag{17}$$

in which the prime superscript denotes the magnitude of S_e or P_e associated with any channel parameter (ψ') having a different value than the constant value of the corresponding parameter (ψ) for which Figures 4 and 6 are applicable. The correction factor η, is presented in Figure 7 for the various channel parameters in terms of the ratio, ψ' / ψ. It can be observed from Figure 7 that the numerical distortion increases when either the channel length, L, or the Manning roughness factor, n, increase; and decreases when either the magnitude of the initial depth of flow, Y_o, or the channel bottom slope, S_o, increase. The channel width, B, was observed to have little or no effect on the magnitude of the numerical distortion. The magnitude of the numerical distortion increases with the distance from the upstream boundary to the channel location in question.

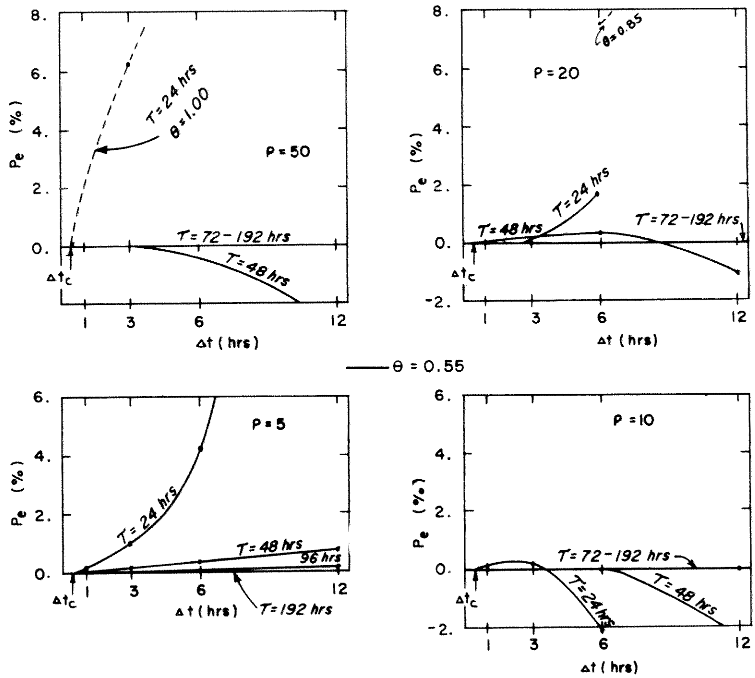

Figure 6. Correlation of P_e error (for the stage at the downstream boundary) with the Δt time step for various upstream boundary hydrographs having $\gamma = 1.2$.

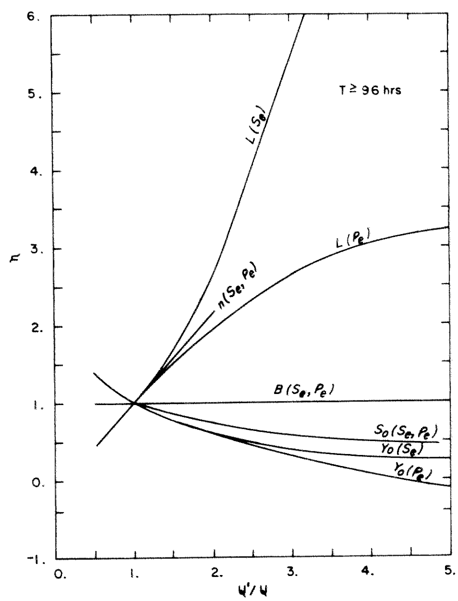

Figure 7. Correction factor, η, for determining the effect of various channel parameters on the numerical distortion (S_e, P_e).

The effects of the channel parameters on the magnitude of the numerical distortion of transients with τ values less than 96 hours were more difficult to summarize as they did not appear to follow a general pattern and, as a result, are not presented herein.

Although only stage hydrographs have been used to illustrate the characteristics of the numerical distortion produced by large Δt time steps, computed discharge hydrographs were subject to numerical distortion of the same order of magnitude.

COMPUTATION TIME

The required computation time on a CDC 6600 computer is shown in Figure 8 for various Δt time steps and two upstream boundary transients. Although the computation times presented in Figure 8 are applicable for 10 Δx sub-reaches, computation times for other N values may be readily determined since the required computation time is directly proportional to N.

It is apparent from Figure 8, that the required computation time is reduced considerably as Δt increases from 0.5 hour to approximately 6 hours and then decreases very little as Δt approaches 12 hours. Since the magnitude of the distortion increases as Δt increases (refer to Figures 4 and 6), Δt time steps in the range of 3 to 6 hours will minimize both the computation time and the numerical distortion.

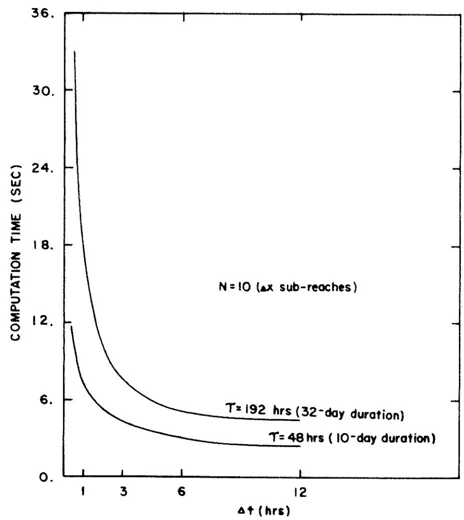

Figure 8. Effect of Δt and τ on required computer time (CDC 6600) for the duration of the transient.

SUMMARY AND CONCLUSIONS

The effects of large time steps in the integration of the implicit finite difference equations of unsteady flow have been investigated for typical single-peak transients at the upstream boundary. The influence of a range of channel parameters has been included in the analysis. The conclusions resulting from the investigation are summarized as follows:

1) Numerical distortion, in the form of dispersion and attenuation of the computed transient, increases as the size of the Δt time step increases;

2) Numerical distortion of the computed transient increases as the θ weighting factor in the implicit difference equations appraoches unity;
3) Numerical distortion, measured by S_e and P_e, is of the order of one percent or less for $\Delta t \leq 12$ hours when the transients at the upstream boundary have a time of rise (τ) greater than approximately 72 hours; this is applicable for $\theta = 0.55$, $L = 100$ miles and $n = 0.03$, and increases as θ, L, and/or n increase;
4) When $\tau > 96$ hours, the magnitude of the numerical distortion is approximately proportional to certain computational, upstream boundary, and channel parameters as follows:

$$[S_e, P_e] \propto \Delta t, \theta, \tau^{-1}, p, n, L, Y_o^{-1}, S_o^{-1}$$

5) The implicit difference equations are more stable for large Δt time steps and relatively rapid transients ($24 \leq \tau \leq 48$ hours) as θ approaches unity; however, the truncation error becomes quite large for Δt much greater than approximately 1 or 2 hours.

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EFFECTS OF TIME STEP SIZE IN IMPLICIT DYNAMIC ROUTING

by

D. L. Fread[1]

ABSTRACT. The effects of the size of the Δt time step used in the integration of the implicit difference equations of unsteady open-channel flow are determined for numerous typical hydrographs with durations in the order of days or even weeks. Truncation errors related to the size of the Δt time step cause a numerical distortion (dispersion and attenuation) of the computed transient. The magnitude of the distortion is related directly to the size of the time step, the length of channel reach, and the channel resistance and inversely to the time of rise of the hydrograph. The type of finite difference expression which replaces spatial derivatives and non-derivative terms in the partial differential equations of unsteady flow has an important influence on the magnitude of the numerical distortion, as well as the numerical stability of the implicit difference equations. Time step sizes in the range of 3 to 6 hrs generally tend to minimize the combination of required computation time and numerical distortion of transients having a time of rise of the order of several days.
(KEY TERMS: open-channel flow, unsteady flow equations, finite differences, implicit method, truncation errors)

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EFFECTS OF TIME STEP SIZE IN IMPLICIT DYNAMIC ROUTING

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INTRODUCTION

Unsteady or transient flow in open channels such as rivers, canals, reservoirs, etc., may be simulated by a mathematical model based on the complete one-dimensional unsteady flow equations which conserve the mass and the momentum of the flow. Analytical solutions to these nonlinear partial differential equations do not exist. However, they may be solved by numerical techniques which use algebraic finite difference equations to approximate the partial differential equations. It is essential to utilize a digital computer to perform the numerous computations required by this solution technique.

Numerous finite difference techniques for numerically integrating the unsteady flow equations have been reported in the literature. They may be categorized into the following four methods of solution:

1. Implicit method [Abbott and Ionescu, 1967; Lai, 1967; Baltzer and Lai, 1968; Amein, 1968; Dronkers, 1969; Amein and Fang, 1970; Strelkoff, 1970; Kamphauis, 1970; Gunaratnam and Perkins, 1970; Contractor and Wiggert, 1971; Fread, 1972];
2. Explicit method [Isaacson, Stoker and Troesch, 1956; Stoker, 1957; Liggett and Woolhiser, 1967; Dronkers, 1969; Garrison, Granju and Price, 1969; Strelkoff, 1970; Strelkoff and Terzidis, 1970];

3. Characteristic method with curvilinear net [Lister, 1960; Amein, 1966; Streeter and Wylie, 1967; Liggett and Woolhiser, 1967; Wylie, 1969; Fread and Harbaugh, 1972]; and

4. Characteristic method with rectangular net [Lister, 1960; Streeter and Wylie, 1967; Baltzer and Lai, 1968; Mozayeny and Song, 1969; Wylie, 1970; Yevjevich and Barnes, 1970].

Of the four methods, the implicit method appears to be best suited for modeling transient flows with durations in the order of days or weeks such as the natural floods occurring in large river systems. The implicit method, unlike the other methods, theoretically does not restrict the size of time step because of the numerical stability characteristics of the finite difference equations. Large time steps can enable the implicit method to be more computationally efficient than the other methods, particularly for long duration transients.

The aim of this paper is to investigate the effect of large time steps on the accuracy of solutions obtained from the unsteady flow equations by the implicit finite difference technique for transient flows of durations in the order of days and weeks.

UNSTEADY FLOW EQUATIONS

The unsteady flow equations, the equation of continuity (conservation of mass) and the equation of motion (conservation of momentum), can be respectively expressed in the divergence form as:

$$\frac{\partial A}{\partial t} + \frac{\partial (AV)}{\partial x} - q = 0 \tag{1}$$

$$\frac{\partial (AV)}{\partial t} + \frac{\partial (AV^2)}{\partial x} + gA\left(\frac{\partial h}{\partial x} + S_f\right) - qv_x = 0 \tag{2}$$

in which the resistance slope, S_f, is given by the Manning equation, i.e.,

$$S_f = n^2|V|V/[2.21(\frac{A}{P})^{4/3}] \tag{3}$$

The terms in the above equations are defined as: x = longitudinal distance along the channel, positive in the downstream direction; t = time; A = cross-sectional area of flow; V = mean velocity of flow across a section, positive in the downstream direction; h = water surface elevation; q = known lateral inflow or outflow per unit length along the channel, positive if inflow; v_x = velocity of lateral flow in the direction of the channel flow; S_f = resistance slope; n = Manning roughness coefficient; P = wetted perimeter of the flow cross section; and g = acceleration of gravity.

A derivation of the equations may be found in several references, e.g., Stoker [1957], Chow [1959] and Strelkoff [1969]. It is assumed in the derivation that the flow is one-dimensional in the sense that flow characteristics such as depth and velocity are considered to vary only in the longitudinal x-direction of the channel. It is further assumed that: 1) the velocity is constant and the water surface is horizontal across any section perpendicular to the longitudinal axis; 2) the flow is gradually varied with hydrostatic pressure prevailing at all points in the flow such that the vertical acceleration of water particles may be neglected; 3) the longitudinal axis of the channel can be approximated by a straight line; 4) the bottom slope of the channel is small; 5) the bed of the channel is fixed, i.e., no scouring or deposition is assumed to occur; 6) the resistance coefficient for steady uniform turbulent flow is considered applicable, and an empirical resistance equation such as the Manning equation describes the resistance effects; and 7) the flow is incompressible and homogeneous in density.

Equations 1 and 2 make up a system of two nonlinear, first order, first degree partial differential equations of the hyperbolic type. They have x and t as independent variables and h and V as dependent variables. The other terms are constants or are functions of independent and/or dependent variables, i.e., A(x,h), S_f(x,h,V), q(x,t), v_x(x,t), n(x,h) and P(x,h).

In order to obtain solutions to the unsteady flow equations, it is necessary to specify boundary and initial conditions. Boundary conditions are conditions specified at fixed values of x for various time. These values include discharge or water surface elevation versus time, or a stage-discharge relation for the upstream and downstream extremities of the channel reach. Initial conditions are conditions specified at fixed values of time at various spatial locations. An initial flow profile for the channel reach may be determined from a backwater computation [Fread and Harbaugh, 1971] and used as an initial condition. Besides boundary and initial conditions, lateral flows, channel geometry and resistance coefficients must be prescribed a priori.

IMPLICIT FINITE DIFFERENCE SOLUTION

Equations 1 and 2 may be approximated by algebraic finite difference equations; and the continuous x-t region in which solutions of h and V are desired can be represented by a rectangular net of discrete points. The net points are defined by the intersection of straight lines drawn parallel to the axes of the x-t region. Lines parallel to the x-axis are time lines and have a spacing of Δt which need not be constant. Lines parallel to the t-axis represent locations along the channel and have a spacing of Δx which need not be constant. Each discrete point may be identified by a double subscript (i,j); the first designates the x-position and the second designates the time line.

In the implicit finite difference solution, the time derivatives are approximated by a forward difference quotient centered between the i^{th} and i+1 points, i.e.,

$$\frac{\partial K}{\partial t} \simeq \frac{K_{i,j+1} + K_{i+1,j+1} - K_{i,j} - K_{i+1,j}}{2\Delta t_j} \tag{4}$$

where K represents any function or variable. The spatial derivatives are approximated by a forward difference quotient positioned between two adjacent time lines according to weighting factors of θ and $(1-\theta)$, i.e.,

$$\frac{\partial K}{\partial x} \simeq \frac{\theta(K_{i+1,j+1} - K_{i,j+1})}{\Delta x_i} + \frac{(1-\theta)(K_{i+1,j} - K_{i,j})}{\Delta x_i} \tag{5}$$

Functions other than derivatives are approximated by using weighting factors similar to Eq. 5. Thus,

$$K \simeq \frac{\theta(K_{i,j+1} + K_{i+1,j+1})}{2} + \frac{(1-\theta)(K_{i,j} + K_{i+1,j})}{2} \tag{6}$$

Upon substituting the finite difference operators defined by Equations 4, 5, and 6 into the unsteady flow Equations 1 and 2, the following implicit difference equations are obtained:

$$\frac{A_{i,j+1} + A_{i+1,j+1} - A_{i,j} - A_{i+1,j}}{2\Delta t_j} + \theta\left[\frac{(AV)_{i+1,j+1} - (AV)_{i,j+1}}{\Delta x_i} - q_{i,j+1}\right]$$

$$+ (1-\theta)\left[\frac{(AV)_{i+1,j} - (AV)_{i+1,j}}{\Delta x_i} - q_{i,j}\right] = 0 \tag{7}$$

$$\frac{(AV)_{i,j+1} + (AV)_{i+1,j+1} - (AV)_{i,j} - (AV)_{i+1,j}}{2\Delta t_j} + \theta\left[\frac{(AV^2)_{i+1,j+1} - (AV^2)_{i,j+1}}{\Delta x_i}\right.$$

$$+ g\frac{(A_{i,j+1} + A_{i+1,j+1})}{2}\left(\frac{h_{i+1,j+1} - h_{i,j+1}}{\Delta x_i} + \frac{S_{f_{i,j+1}} + S_{f_{i+1,j+1}}}{2}\right) - (qv_x)_{i,j+1}$$

$$+ (1-\theta)\left[\frac{(AV^2)_{i+1,j} - (AV^2)_{i,j}}{\Delta x_i} + g\frac{(A_{i,j} + A_{i+1,j})}{2}\left(\frac{h_{i+1,j} - h_{i,j}}{\Delta x_i} + \right.\right.$$

$$\left.\left.\frac{S_{f_{i,j}} + S_{f_{i+1,j}}}{2}\right) - (qv_x)_{i,j}\right] = 0 \tag{8}$$

A weighting factor of $\theta = 1$ yields the fully implicit scheme used by Baltzer and Lai [1968]. A weighting factor of $\theta = 1/2$ produces the "box" scheme used by Issacson [1966], Amein [1968], Amein and Fang [1970], and Contractor and Wiggert [1971].

Equations 7 and 8 form a system of two algebraic equations which are nonlinear with respect to the unknowns, the values of h and V at the net points (i,j+1) and (i+1,j+1). The terms A and S_f are known functions of h and/or V. The terms associated with the net points (i,j) and (i+1,j) are known from either the initial conditions or previous computations.

The two equations cannot be solved for the unknowns since there are two more unknowns than equations; however, by considering all N number of points along the x-axis simultaneously, a solution may be obtained. In this way, a total of (2N-2) equations with 2N unknowns may be formulated by applying Equations 7 and 8 recursively to the (N-1) rectangular grids along the x-axis. The boundary conditions at the upstream and downstream extremities of the channel reach provide two additional equations which are necessary for the system of equations to be sufficiently proposed to yield a solution. The resulting system of 2N nonlinear equations with 2N unknowns must be solved by an iterative procedure. A functional iterative process, called Newton-Raphson Iteration [Crandall, 1956; Amein and Fang, 1970], is used to solve the nonlinear system. The iterative process converges to a solution of acceptable accuracy at a quadratic rate; this may be improved by using parabolic extrapolation to obtain the first approximation of the solution from solutions determined at previous times. The coefficient matrix of the linearized system of equations has a banded structure which lends itself to very efficient solution algorithms, e.g., [Fread, 1970].

STABILITY OF THE IMPLICIT EQUATIONS

The solution of a system of finite difference equations requires that numerical errors of round-off, introduced in the computational procedure, not be amplified into an unlimited error. The stability of the nonlinear difference Equations 7 and 8 can be investigated by a Fourier analysis of the error propagation properties of linearized forms of the difference equations. This stability analysis, known as the von Neumann method [O'Brien, Hyman, and Kaplan, 1951] has been used by various investigators, e.g. Abbott and Ionescu [1967] and Leendertse [1967], to show that, in general, an implicit finite difference formulation of the unsteady flow equations is unconditionally stable for any ratio of $\Delta x/\Delta t$ when the θ weighting factor is restricted to the range, $1/2 \leq \theta \leq 1$. Thus, according to this stability analysis, the stability of the implicit method does not depend on the ratio, $\Delta x/\Delta t$, as do the explicit and characteristic methods. However, the inability to include in the stability analysis the nonlinearities of Equations 7 and 8, as well as the effects of boundary conditions, causes the von Neumann technique to be heurisitic and somewhat inconclusive. Under certain conditions, the implicit difference equations have been observed to exhibit instabilities [Liggett and Woolhiser, 1967; Baltzer and Lai, 1972]. In this investigation, numerical instabilities were encountered for certain upstream boundary hydrographs and Δt time steps; this will be discussed later.

ACCURACY OF THE IMPLICIT EQUATIONS

Solutions obtained from the implicit difference Equations 7 and 8 have been mathematically shown to converge to the true solutions of the partial differential Equations 1 and 2 as Δx and Δt approach zero [Abbott and Inoescu, 1967 and Leendertse, 1967]. Thus, if channel length and the irregularity of channel geometry are used to select Δx, the accuracy of the implicit difference solution decreases as the size of the time step increases.

Truncation errors, related to the magnitude of the time step, arise during the integration of the implicit difference equations. The truncation errors distort the solution via numerical dispersion and attenuation of the computed transient. Henceforth, the truncation error in the solution will be referred to as "numerical distortion." Also, as will be shown later, the characteristics of the discharge hydrograph at the upstream extremity of the channel reach have a significant effect on the accuracy of the solution.

The characteristics of the numerical distortion are investigated herein via numerical experiments in which Equations 7 and 8 are applied to upstream boundary transients described by the following four-parameter, Pearson Type III distribution:

$$Q(t) = Q_o[1+(\rho-1)\ (\frac{t}{\tau})^{(\frac{1}{\gamma-1})}\ e^{\rho(1-\frac{t}{\tau})}] \tag{9}$$

in which

$$\rho = Q_{max}/Q_o \tag{10}$$

$$\gamma = \tau_g/\tau \tag{11}$$

The terms in the above equations are defined as follows: $Q(t)$ = discharge at any time (t), Q_o = initial steady discharge as computed by the Manning equation, Q_{max} = maximum discharge at the upstream boundary during the transient flow condition, τ = time of occurrence of Q_{max}, τ_g = time associated with the center of gravity of the upstream hydrograph, ρ = hydrograph amplification coefficient, and γ = a skewness coefficient of the upstream hydrograph.

The downstream boundary condition is specified by the following implicit stage-velocity relationship which is corrected for transient effects:

$$V = \frac{1.486}{n}\ (\frac{A}{p})^{2/3} S_f^{\ 1/2} \tag{12}$$

in which

$$S_f = (qv_x - \frac{\partial(AV)}{\partial t} - \frac{\partial(AV^2)}{\partial x} - \frac{\partial h}{\partial x})/gA \tag{13}$$

This boundary condition allows the transient to pass the downstream extremity of the channel reach with no numerical reflection.

The primary objective of this investigation is to study the effect of the size of the time step on the solutions of the implicit difference equations. Therefore, selected parameters describing the physical characteristics of the channel reach are held constant throughout the study except in special instances where a single parameter is perturbed in order to determine its effect on the results. The selected channel parameters are as follows: channel reach length (L) = 100 miles; channel bottom slope (S_0) = 1/5280 ft per ft; Manning roughness coefficient (n) = 0.03; wide rectangular cross-section with surface width (B) = 2000 ft; number (N) of Δx sub-reaches = 10, and initial depth of flow (Y_0) = 5 ft. Convergence criteria for h and V in the iterative solution were chosen as : $|h^{k+1}-h^k| \leq 1x10^{-6}$ and $|V^{k+1} - V^k| \leq 1x10^{-6}$, where the superscript k denotes the number of iterations.

The effect of the magnitude of the time step on the accuracy of the computed solutions is determined by systematically increasing the time step from Δt_c, a relatively small value in the order of minutes, to a relatively large value of 12 hrs. The Δt_c time step is the maximum size time step that can be used in an explicit method; it is computed from the Courant condition [Stoker, 1957; Strelkoff, 1970] which insures numerical stability when friction effects are relatively small:

$$\Delta t_E \leq \Delta x_i/[|V_i| + (gA/B)_i^{1/2}] \quad . . . \text{(minimum for i=1 . . . . N)} \quad (14)$$

The stage hydrographs obtained using Δt_c in Equations 7 and 8 are considered the standards to which the solutions computed with Δt time steps of 1, 3, 6 and 12 hrs are compared.

This follows the approach used by Abbott and Ionescu [1967] for testing the effect of the magnitude of Δt in direct finite difference approximations of the unsteady flow equations. The fact that the truncation error is a minimum when the time step is Δt_c follows from a Taylor series analysis of Equations 7 and 8, as well as from the fact that as Δt increases beyond Δt_c the response properties of the computational system depart from those of the physical system.

Deviations from the standard hydrographs are measured by the following relative root mean square error (S_e) and relative error of the peak (P_e) of the hydrographs:

$$S_e = \frac{100\left[\sum_{i=1}^{i=n} (y_i - ys_i)^2\right]^{1/2}}{n^{1/2} ys_p} \qquad (15)$$

$$P_e = 100\ (1 - y_p/ys_p) \qquad (16)$$

in which n = total number of hydrograph values being compared, y_i = stage value computed with a particular Δt time step, ys_i = stage value computed with a Δt_c time step, y_p = maximum (peak) value of Y_i, and ys_p = maximum value of ys_i.

Figures 1 and 2 illustrate typical numerical distortions of the computed hydrographs at the downstream boundary for two variations in the upstream boundary condition. In Figure 1, the time of rise (τ) is 48 hrs, while in Figure 2, τ is 120 hrs. The hydrographs obtained with a time step of 12 hrs differ from those computed with a time step of 0.5 hr. The rising limb of the former occurs earlier than the latter, while the falling limb is delayed and the peak is attenuated. The distortion is more pronounced in Figure 1 than in Figure 2 for the same values of Δt and θ. Also, for a single τ value, the distortion is significantly greater for θ = 1 than for θ = 0.55.

A quantitative evaluation of the numerical distortion, in terms of S_e and P_e, is shown in Figure 3. The influence of θ and τ on the degree of distortion is significant. This was also observed for other test hydrographs. Thus, it may be concluded that the lower range of allowable θ values minimizes the distortion (dispersion and attenuation) which results from the use of large time steps in the integration of the implicit difference equations. Also, the degree of distortion becomes less as the time of rise of the input hydrograph increases. Several correlations of S_e with the size of the Δt time step are shown in Figure 4. The correlations are given for various τ and ρ values of the upstream boundary hydrograph. The S_e error is associated with the stage hydrographs computed at the downstream boundary of the 100 mile channel reach described previously.

An examination of Figure 4 yields the following information concerning the numerical distortion resulting from the use of Δt time steps considerably larger than those determined from the Courant Condition (Eq. 14):

1) The magnitude of S_e increases with the size of the Δt time step;
2) As τ, the time of rise of the upstream hydrograph increases, the slopes of the (S_e, Δt) curves decrease;
3) The magnitude of S_e is less than 1% for $\tau \geq 96$ hrs and $\Delta t \leq 12$ hrs.

The solid curves in Figure 4 are applicable for a θ of 0.55, a value chosen so as to minimize numerical distortion while conservatively insuring theoretical stability of the computations. The dashed portion of the curves are applicable to θ values greater than 0.55 which are required for numerical stability since lesser values of θ cause instabilities to arise in the iterative solution of the nonlinear difference equations. The selected θ values are optimal in that the magnitude of numerical distortion is minimized while numerical stability is achieved. The optimal θ values vary with Δt and

τ. An illustration of the variation with Δt and τ is given in Figure 5 for a ρ of 20. By inspecting Figures 4 and 5, it can be seen that the tendency for stable numerical computations decreases with increasing values of Δt and with decreasing values of θ and τ.

The effect of the Δt time step size on the attenuation of the computed stage hydrographs at the downstream boundary is presented in Figure 6 for various combinations for τ and ρ. In Figure 6, P_e is negligible for τ values greater than 48 hrs; however, P_e can be significant for $\Delta t > 3$ hr when $\tau \leq 48$ hrs.

The results presented thus far are applicable for the constant channel parameters selected previously. In order to determine if the numerical distortion resulting from large time steps is sensitive to the values of the channel parameters, these are perturbed and the resulting effects on S_e and P_e are observed. The observed effects may be summarized by the following approximation:

$$(S'_e, P'_e) \simeq \eta . (S_e, P_e) \tag{17}$$

in which the prime superscript denotes the magnitude of S_e or P_e associated with any channel parameter (ψ') having a different value than the constant value of the corresponding parameter (ψ) for which Figures 4 and 6 are applicable. The correction factor η, is presented in Figure 7 for the various channel parameters in terms of the ratio, ψ'/ψ. It can be observed from Figure 7 that the numerical distortion increases when either the channel length, L, or the Manning roughness factor, n, increase; and decreases when either the magnitude of the initial depth of flow, Y_o, or the channel bottom slope, S_o, increase. The channel width, B, was observed to have little or no effect on the magnitude of the numerical distortion. The magnitude of the numerical distortion increases with the distance from the upstream boundary to the channel location in question.

The effects of the channel parameters on the magnitude of the numerical distortion of transients with τ values less than 96 hours were more difficult to summarize as they did not appear to follow a general pattern and, as a result, are not presented herein.

Although only stage hydrographs have been used to illustrate the characteristics of the numerical distortion produced by large Δt time steps, computed discharge hydrographs were subject to numerical distortion of the same order of magnitude.

COMPUTATION TIME

The required computation time on a CDC 6600 computer is shown in Figure 8 for various Δt time steps and two upstream boundary transients. Although the computation times presented in Figure 8 are applicable for 10 Δx sub-reaches, computation times for other N values may be readily determined since the required computation time is directly proportional to N.

It is apparent from Figure 8, that the required computation time is reduced considerably as Δt increases from 0.5 hr to approximately 6 hrs and then decreases very little as Δt approaches 12 hrs. Since the magnitude of the distortion increases as Δt increases (refer to Figures 4 and 6), Δt time steps in the range of 3 to 6 hrs will minimize both the computation time and the numerical distortion.

SUMMARY AND CONCLUSIONS

The effects of large time steps in the integration of the implicit finite difference equations of unsteady flow have been investigated for typical single-peak transients at the upstream boundary. The influence of a range of channel parameters has been included in the analysis. The conclusions resulting from the investigation are summarized as follows:

1) Numerical distortion, in the form of dispersion and attenuation of the computed transient, increases as the size of the Δt time step increases;

2) Numerical distortion of the computed transient increases as the θ weighting factor in the implicit difference equations approaches unity;

3) Numerical distortion, measured by S_e and P_e, is of the order of one percent or less for $\Delta t \leq 12$ hrs when the transients at the upstream boundary have a time of rise (τ) greater than approximately 72 hrs; this is applicable for $\theta = 0.55$, L = 100 miles and n = 0.03, and increases as θ, L, and/or n increase;

4) When $\tau \geq 96$ hrs, the magnitude of the numerical distortion is approximately proportional to certain computational, upstream boundary, and channel parameters as follows:

$$[S_e, P_e] \propto \Delta t, \theta, \tau^{-1}, \rho, n, L, Y_0^{-1}, S_0^{-1}$$

5) The implicit difference equations are more stable for large Δt time steps and relatively rapid transients ($24 \leq \tau \leq 48$ hrs) as θ approaches unity; however, the truncation error becomes quite large for Δt much greater than approximately 1 or 2 hrs.

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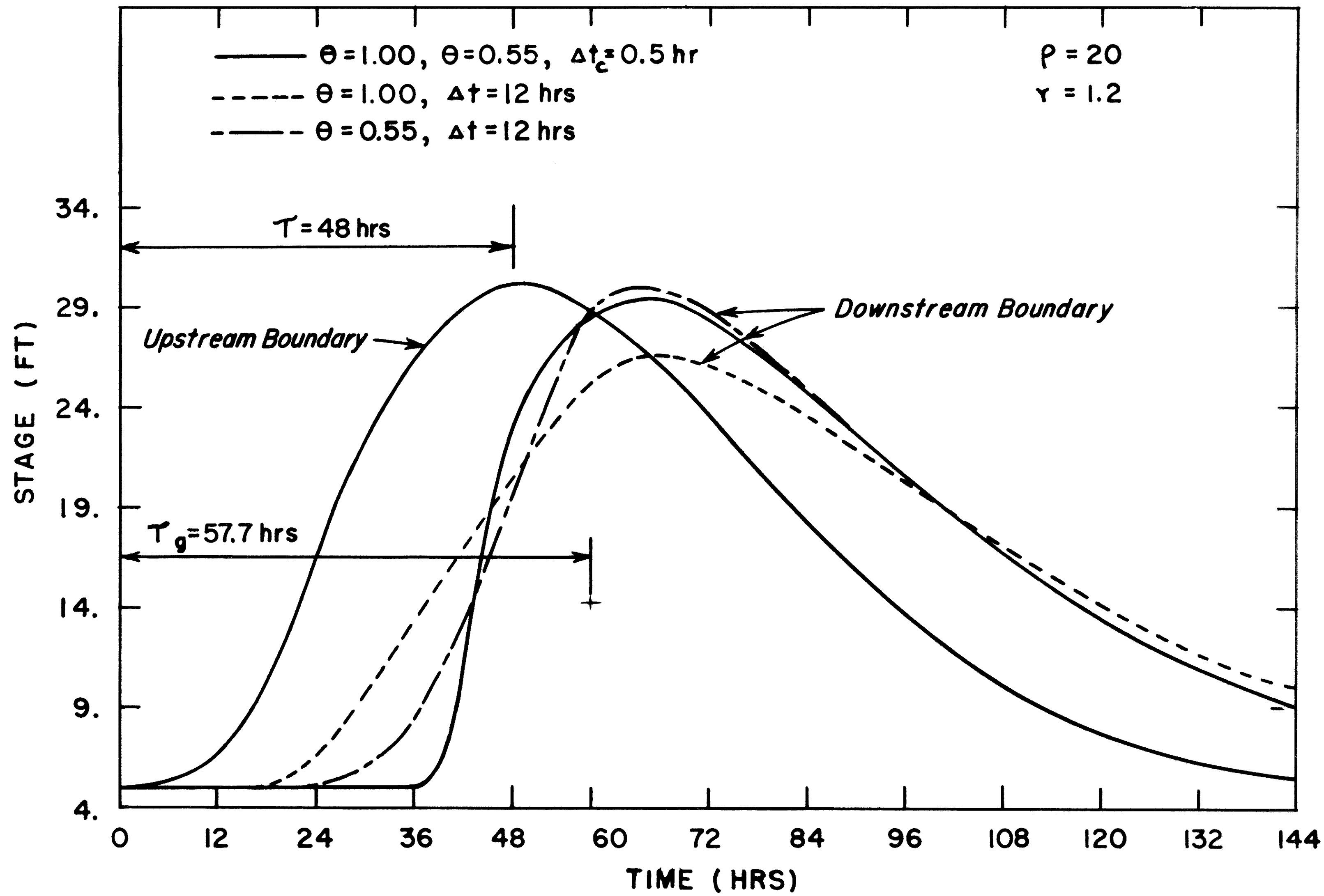

Fig. 1. Distortion of computed downstream stage hydrograph for large Δt steps when θ is varied and τ = 48 hours.

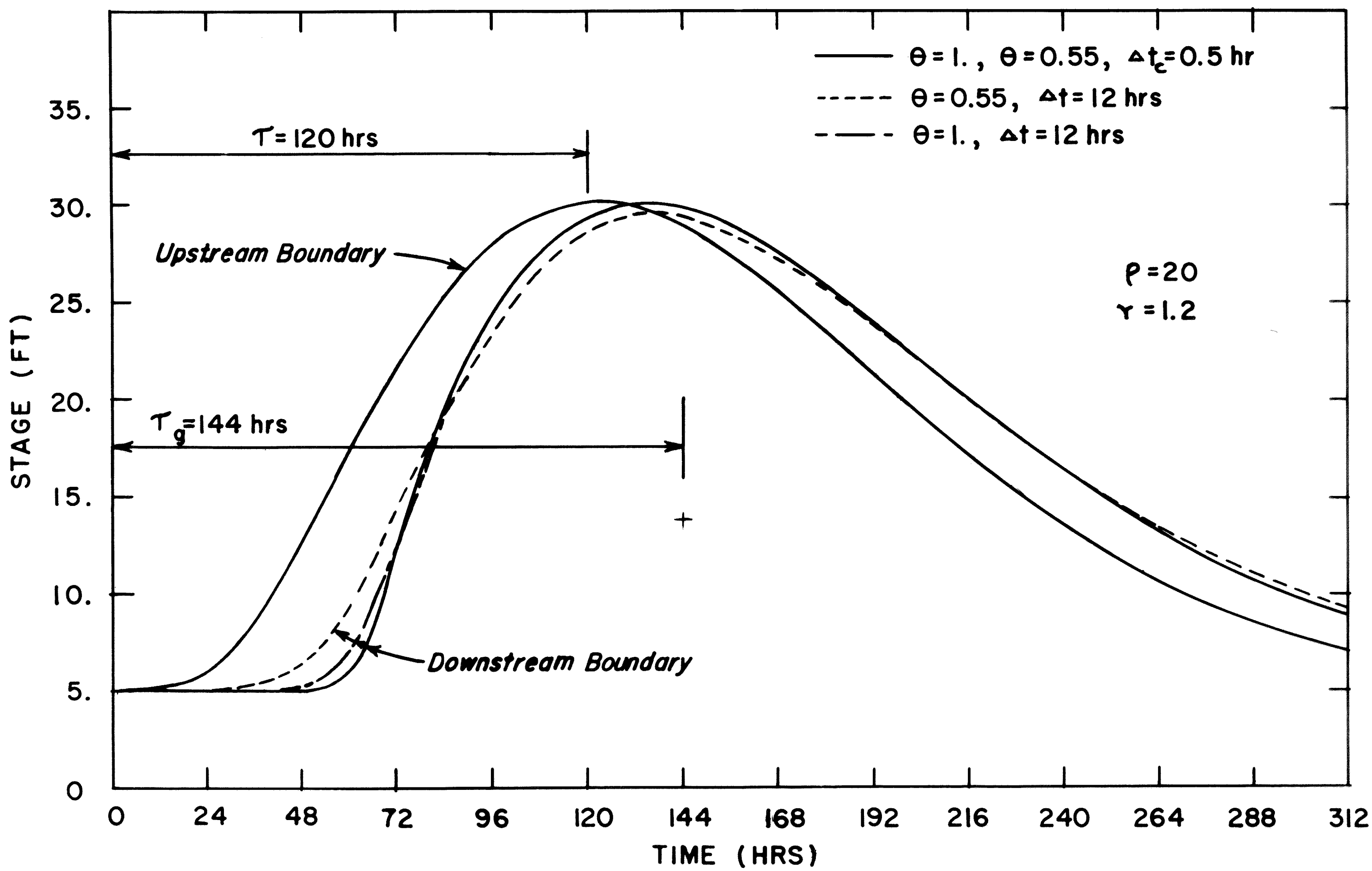

Fig. 2. Distortion of computed downstream stage hydrograph for large Δt steps when θ is varied and τ = 120 hours.

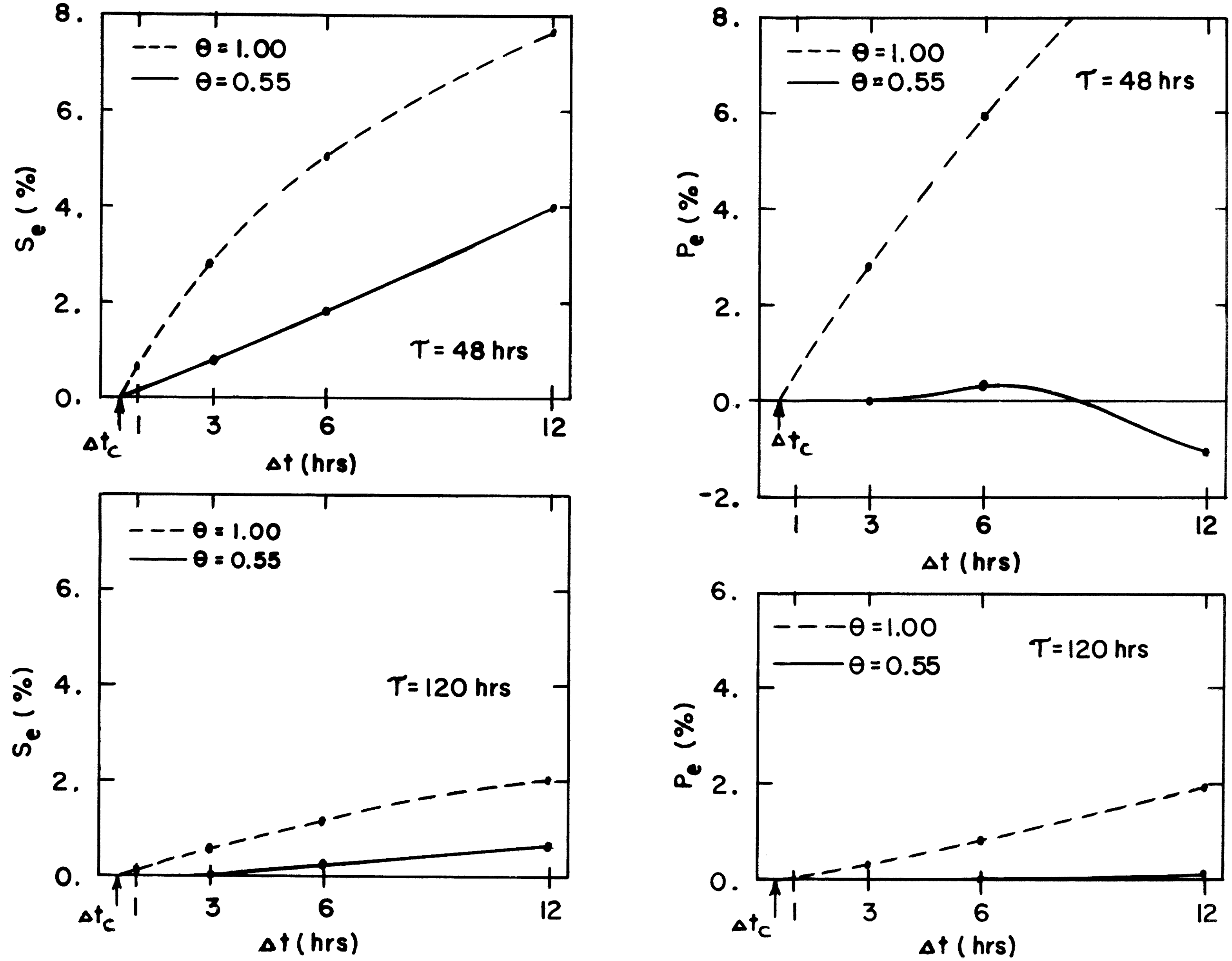

Fig. 3. Effect of θ and τ on the distortion of the computed stage hydrograph at the downstream boundary for various Δt time steps having $\rho = 20$ and $\gamma = 1.2$.

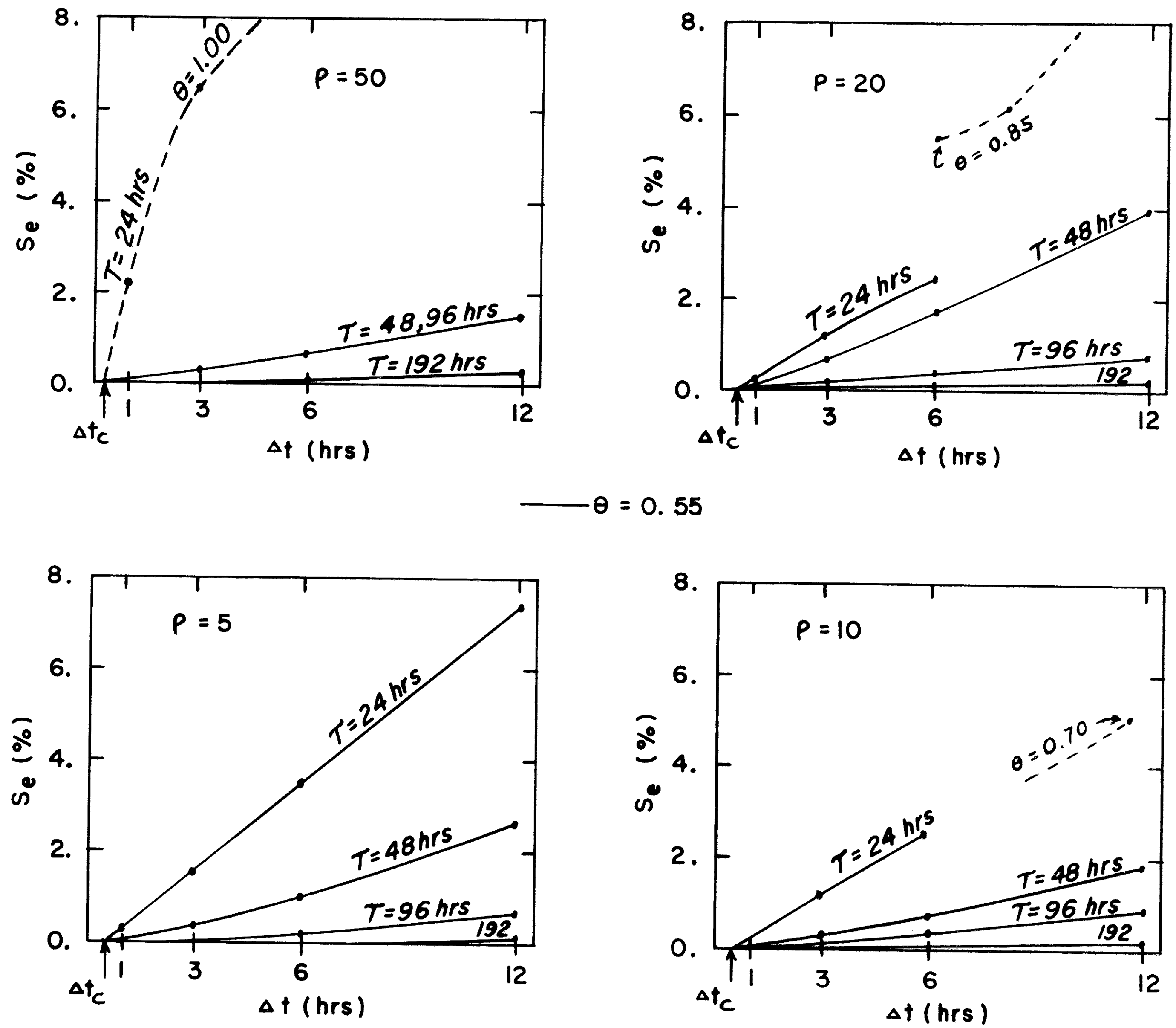

Fig. 4. Correlation of S_e error (for the stage at the downstream boundary) with the Δt time step for various upstream boundary hydrographs having γ=1.2.

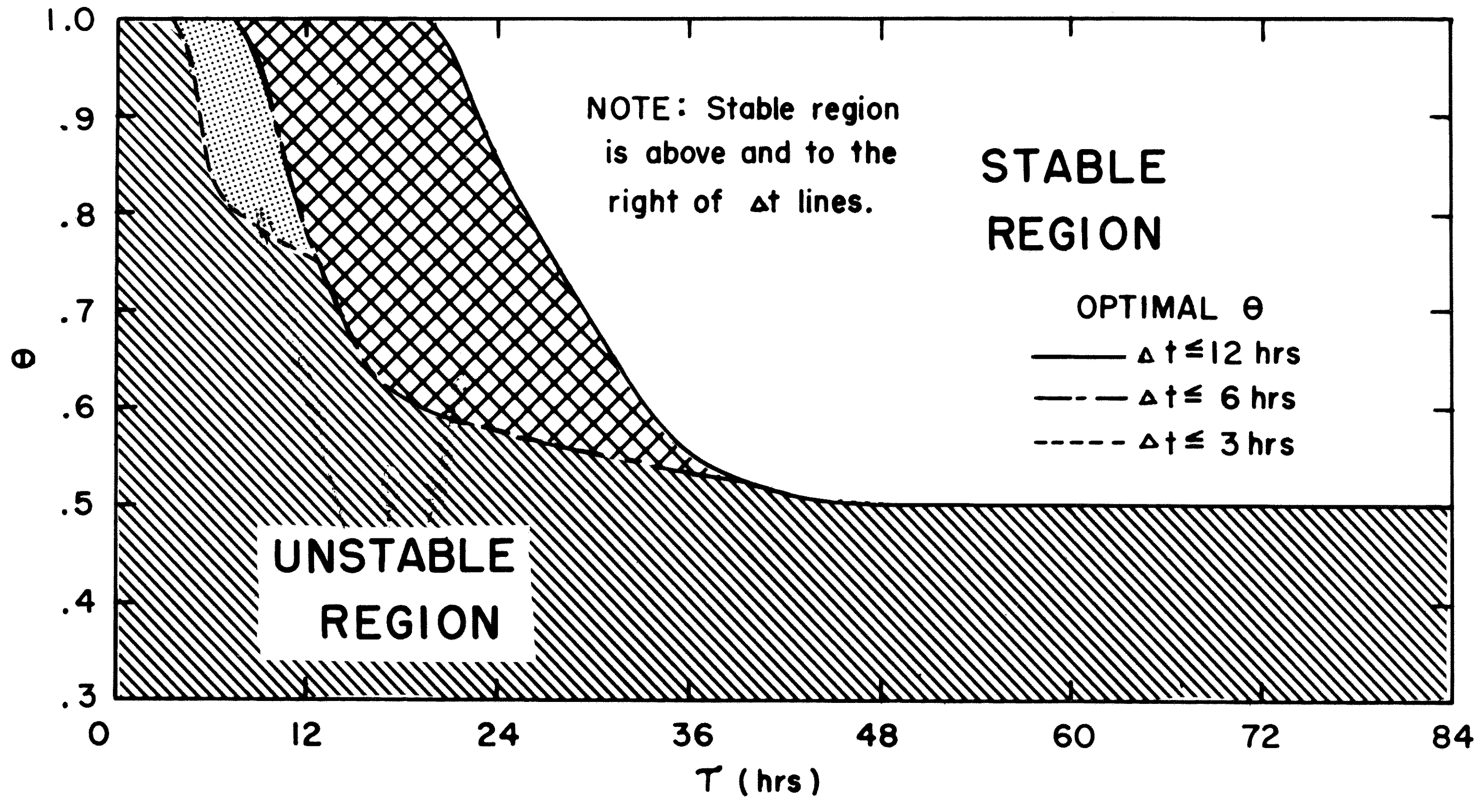

Fig. 5. Optimal θ for numerical stability and minimum distortion for various τ, Δt, and ρ=20, γ=1.2

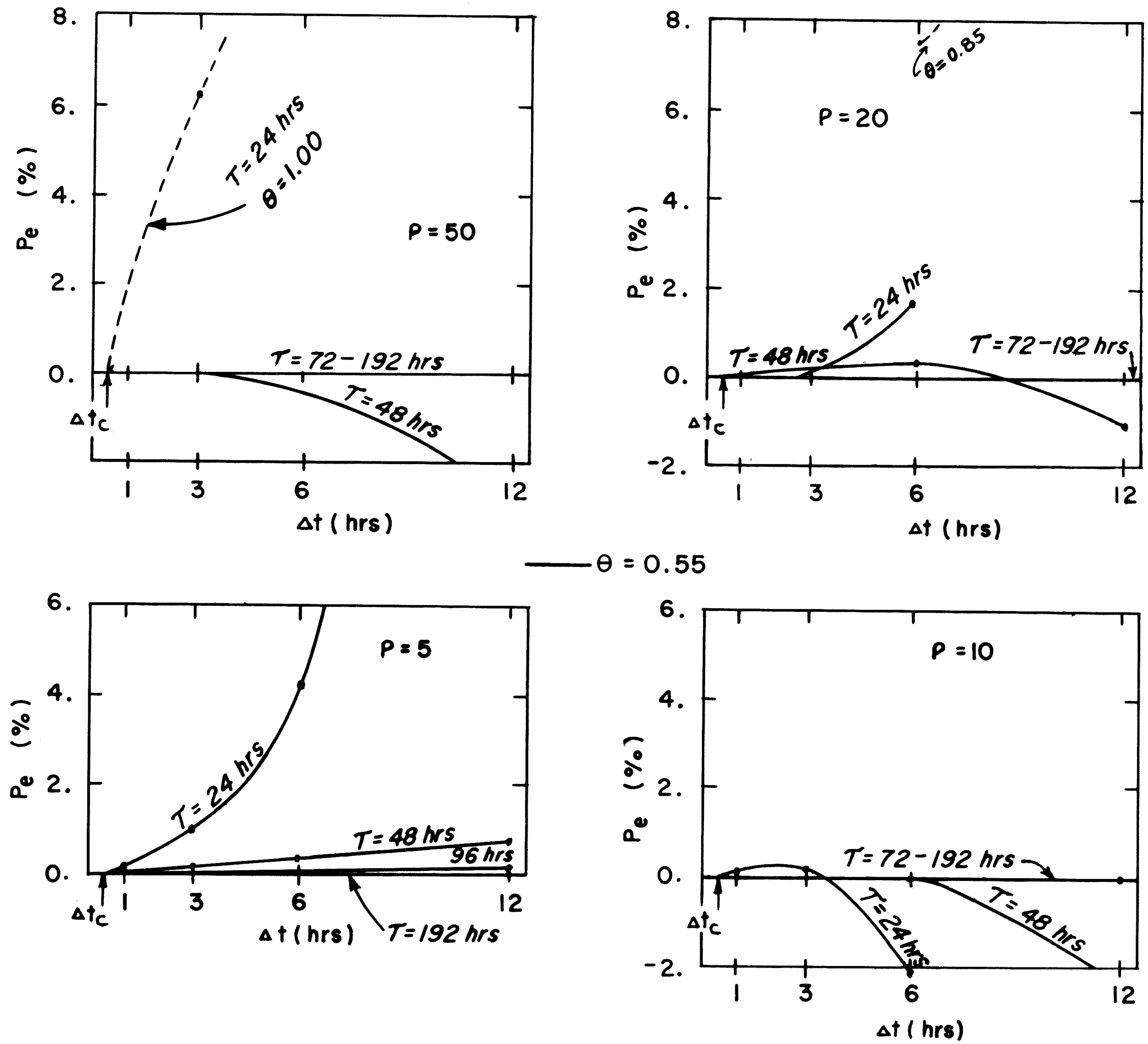

Fig. 6. Correlation of P_e error (for the stage at the downstream boundary) with the Δt time step for various upstream boundary hydrographs having γ = 1.2.

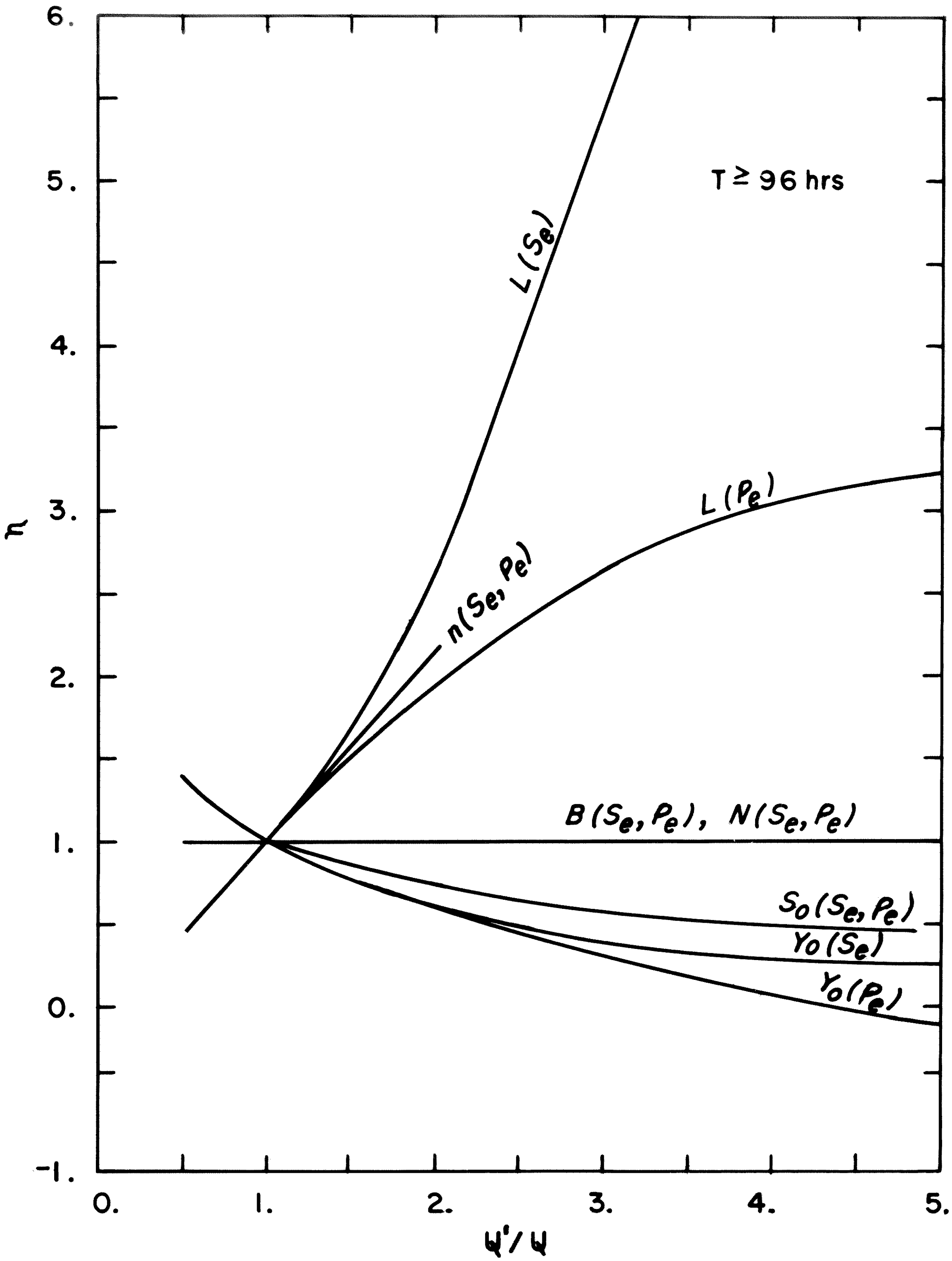

Fig. 7. Correction factor, η, for determining the effect of various channel parameters on the numerical distortion (S_e, P_e)

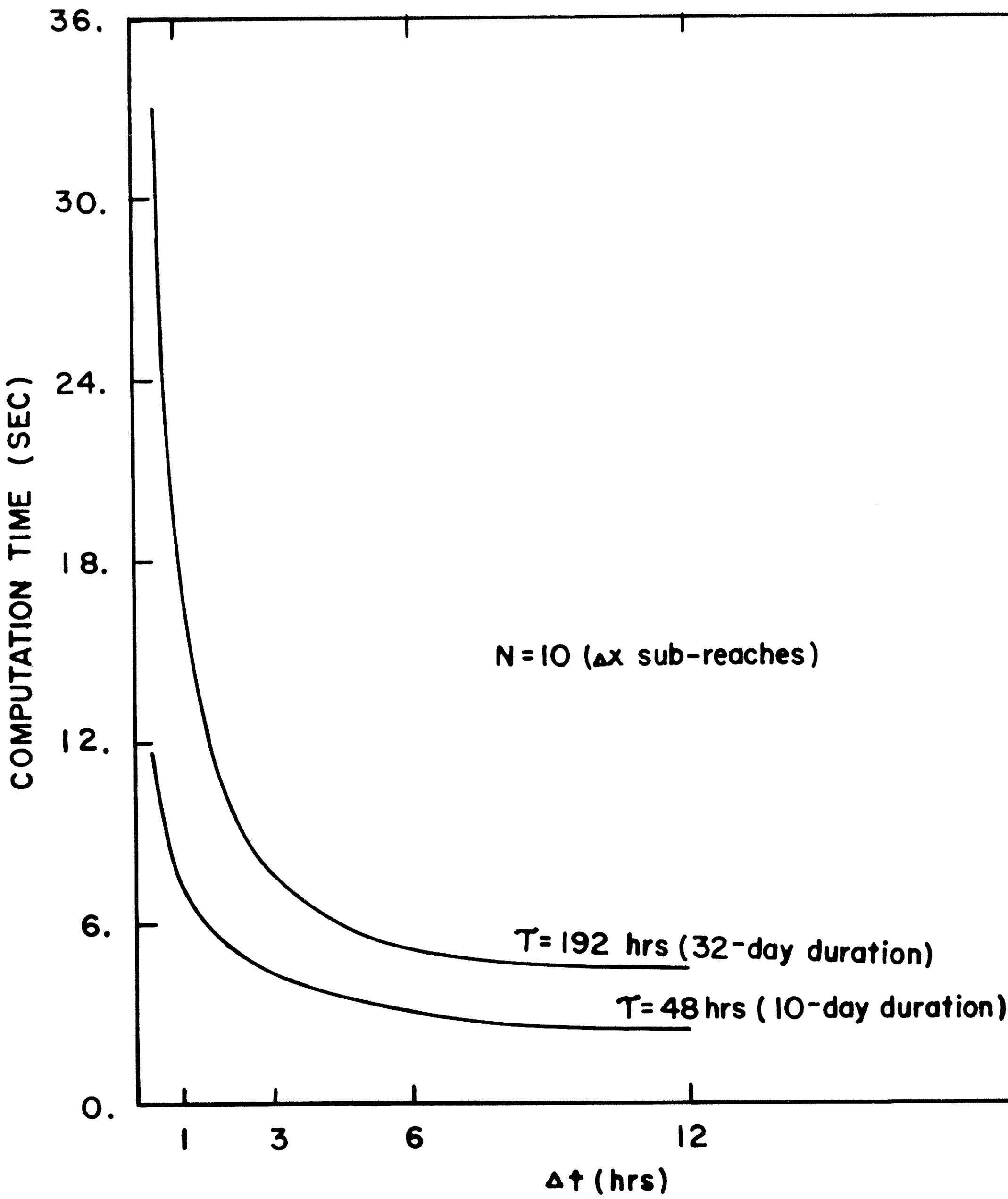

Fig. 8. Effect of Δt and τ on required computer time (CDC 6600) for the duration of the transient.